Tripartite-to-Bipartite Entanglement Distillation by Stochastic Local Operations and Classical Communication

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Outline

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Introduction

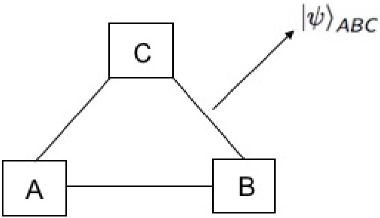
Characterization of the asymptotic tripartite-to-bipartite entanglement distillation by SLOCC

Conclusion and discussion

Introduction

General settings

Preparation: Alice, Bob and Charlie, jointly possess a tripartite state $|\psi\rangle_{ABC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Let $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.



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Task: Distill the maximally entangled state $|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle_{AB}$ between Alice and Bob using stochastic local operations and classical communication (SLOCC).

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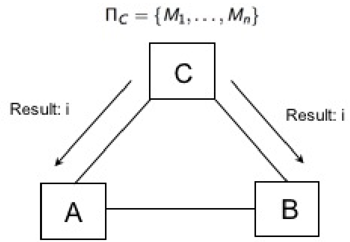

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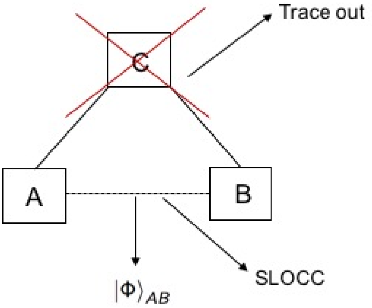

Tripartite-to-bipartite SLOCC convertibility

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Lemma (Chitambar, Duan, Shi, PRA, 2010)

$$|\psi\rangle_{ABC} \xrightarrow{\text{SLOCC}} |\phi\rangle_{AB} \iff \begin{array}{c} \exists\, |\phi'\rangle_{AB} \in \mathrm{supp}(\mathrm{Tr}_C(|\psi\rangle\langle\psi|_{ABC})), \\ \mathrm{Sch}(|\phi'\rangle_{AB}) \geq \mathrm{Sch}(|\phi\rangle_{AB}). \end{array}$$

Tripartite-to-bipartite SLOCC convertibility

Define

$$\mathrm{msrk}(\psi_{ABC}) = \max\{\mathrm{Sch}(|\phi'\rangle_{AB}) : |\phi'\rangle_{AB} \in \mathrm{supp}(\mathrm{Tr}_C(|\psi\rangle\langle\psi|_{ABC}))\}.$$

$\mathrm{msrk}(\psi_{ABC})$ is the largest Schmidt rank of bipartite states which can be obtained from $|\psi\rangle_{ABC}$ by SLOCC.

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Corollary

$|\psi\rangle_{ABC}$ can be transformed to the maximally entangled state $|\Phi\rangle_{AB}$ if and only if $\mathrm{msrk}(\psi_{ABC}) = d$.

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Asymptotic transformation

Now assume Alice, Bob and Charlie, jointly possess many copies of $|\psi\rangle_{ABC}$. Consider whether $|\psi\rangle_{ABC}$ can be transformed to the bipartite maximally entangled state $|\Phi\rangle_{AB}$ by SLOCC asymptotically.

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Mathematically, define the asymptotic maximal Schmidt rank of a tripartite state $|\psi\rangle_{ABC}$:

$$\mathrm{msrk}^{\infty}(\psi_{ABC}) = \lim_{n\to\infty} \sqrt[n]{\mathrm{msrk}(\psi_{ABC}^{\otimes n})}.$$

We need to characterize what kind of states $|\psi\rangle_{ABC}$ satisfy

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It is difficult to calculate the $\mathrm{msrk}^{\infty}(\cdot)$ for arbitrary $|\psi\rangle_{ABC}$.

Isomorphism between bipartite vector spaces and linear operator spaces

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The isomorphism

$$\mathrm{V} : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{L}(\mathcal{H}_B, \mathcal{H}_A),$$

is defined as:

$$\mathrm{V}(|i\rangle \otimes |j\rangle) = |i\rangle \langle j| \,.$$

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Given a bipartite pure state $|\phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have

$$\mathrm{Sch}(|\phi\rangle_{AB}) = \mathrm{rank}(\mathrm{V}(|\phi\rangle_{AB})).$$

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For a tripartite state $|\psi\rangle_{ABC}$, denote

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We have

$$\mathrm{mrk}(M(\psi_{ABC})) = \max\{\mathrm{rank}(S) : S \in M(\psi_{ABC})\} = \mathrm{msrk}(\psi_{ABC}).$$

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Thus we have the following corollary:

Corollary

$|\psi\rangle_{ABC} \xrightarrow{\mathrm{SLOCC}} |\Phi\rangle_{AB}$ $\iff$ *There exists a full-rank matrix in $M(\psi_{ABC})$.*

Outline

Classification of matrix space

Before presenting our main result, we introduce a method to classify matrix spaces based on whether they have full-rank matrices or not. This method can be derived from the bipartite perfect matching problem:

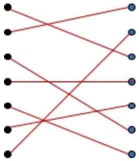

Figure: Classical bipartite perfect matching.

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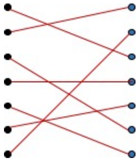

Figure: Classical bipartite perfect matching.

Vertices $\rightarrow$ vectors;
Edges $\rightarrow$ matrices;

Perfect matchings $\rightarrow$ full-rank matrices;

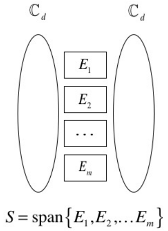

Figure: Quantum bipartite perfect matching.

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Shrunk set and shrunk subspace

Lemma (Hall, JLMS, 1935)

$\nexists$ *Shrunk set*
$\iff$
$\exists$ *perfect matching.*

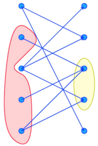

Figure: Shrunk set.

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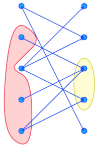

Figure: Shrunk set.

Definition

Given a matrix space $\mathcal{S} \leq \mathcal{M}(d)$, a vector space $U \leq \mathbb{C}_d$ is called a shrunk subspace of $\mathcal{S}$, if $\dim(U) > \dim(\mathcal{S}(U))$, where $\mathcal{S}(U) = \text{span}\{\cup_{E \in \mathcal{S}} EU\}$.

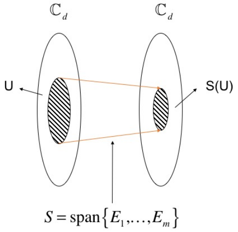

Figure: Shrunk subspace.

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However, there also exist matrix spaces without full-rank matrices or shrunk subspaces. For instance, the 3×3 skew-symmetric matrix space $\left\{ \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}$.

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Non-singular space Matrix spaces which have full-rank matrices;

Shrinking space Matrix spaces Which have shrunk subspaces;

Exceptional space Matrix spaces without full-rank matrices or shrunk subspaces.

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Define the asymptotic maximal rank of a matrix space $\mathcal{S} \leq \mathcal{M}(d)$:

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$$\{|\psi\rangle_{ABC} : \mathrm{msrk}^{\infty}(\psi_{ABC}) = d\} \cong \{\mathcal{S} \leq \mathcal{M}(d) : \mathrm{mrk}^{\infty}(\mathcal{S}) = d\}$$

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Theorem

Given a matrix space $\mathcal{S} \le \mathcal{M}(d)$. $\mathrm{mrk}^{\infty}(\mathcal{S}) = d$ if and only if $\mathcal{S}$ does not have a shrunk subspace.

Proof outline: $\mathcal{S}$ is exceptional

- By the results introduced in [Fortin & Reutenauer, 2004], for an exceptional space $\mathcal{S} \leq \mathcal{M}(d)$, its maximal rank satisfies:

$$\frac{1}{2}d \leq \mathrm{mrk}(\mathcal{S}) \leq d.$$

- Given two exceptional spaces $\mathcal{S}_1, \mathcal{S}_2 \leq \mathcal{M}(d)$, $\mathcal{S}_1 \otimes \mathcal{S}_2$ is also exceptional in $\mathcal{M}(d^2)$. This can be proved by using results in [Ivanyos et al., Computation Complexity, 2016].
- Therefore, for exceptional space $\mathcal{S} \leq \mathcal{M}(d)$,

$$\frac{1}{2}d^n \leq \mathrm{mrk}(\mathcal{S}^{\otimes n}) \leq d^n.$$

Then we can derive for exceptional space $\mathcal{S} \leq \mathcal{M}(d)$,

$$\mathrm{mrk}^{\infty}(\mathcal{S}) = d.$$

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$$\left[\begin{array}{c|c} A_{p\times q} & B_{p\times(d-q)} \\ \hline C_{(d-p)\times q} & \mathbf{0}_{(d-p)\times(d-q)} \end{array}\right]_{d\times d},$$

where p, q are some parameters determined by $\mathcal{S}$ satisfy $p + q < d$.

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where p, q are some parameters determined by $\mathcal{S}$ satisfy $p + q < d$. Define the following matrix space:

$$\begin{aligned}\mathcal{A}(p,q,d) := \mathrm{span}\{&\{|i\rangle\langle j| : 1 \leq i \leq p,\ 1 \leq j \leq d\} \\ &\cup\{|i\rangle\langle j| : p+1 \leq i \leq d,\ 1 \leq j \leq q\}\},\end{aligned}$$

where (p, q, d) are parameters determined by $\mathcal{S}$ satisfying $p + q < d$.

Proof outline: $\mathcal{S}$ is shrinking

Clearly, $\mathcal{A}(p, q, d)$ is also shrinking and $P\mathcal{S}Q \leq \mathcal{A}(p, q, d)$ for some invertible P and Q. Moreover, we have

$$\mathrm{mrk}(\mathcal{S}) \leq \mathrm{mrk}(\mathcal{A}(p, q, d)),$$

and

$$\mathrm{mrk}^{\infty}(\mathcal{S}) \leq \mathrm{mrk}^{\infty}(\mathcal{A}(p, q, d)).$$

Proof outline: $\mathcal{S}$ is shrinking

Theorem

Given $p, q, d \in \mathbb{N}$ *satisfy* $p + q < d$, *let* $p' = \frac{p}{d}$, $q' = \frac{q}{d}$ *and* $\alpha = \frac{\log_2(1-q') - \log_2(p')}{\log_2((1-p')(1-q')) - \log_2(p'q')}$. *Denote* $D(a||b) = a\log_2\frac{a}{b} + (1-a)\log_2\frac{1-a}{1-b}$ *to be the relative entropy of two binary distributions. Then we have:*

$$\mathrm{mrk}^{\infty}(\mathcal{A}(p,q,d)) = d\max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\}.$$

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$$\mathrm{mrk}^{\infty}(\mathcal{A}(p,q,d)) = d \max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\}.$$

It is easy to derive $q' < \alpha < 1 - p'$ by the definition. Then we know $2^{-D(1-\alpha||p')} < 1$ and $2^{-D(\alpha||q')} < 1$ for any (p, q, d) satisfy $p + q < d$. Thus for any shrinking matrix space $\mathcal{S} \leq \mathcal{M}(d)$, we have

$$\lim_{n\to\infty} \sqrt[n]{\mathrm{mrk}(S^{\otimes n})} \leq \lim_{n\to\infty} \sqrt[n]{\mathrm{mrk}(\mathcal{A}(p,q,d)^{\otimes n})} < d.$$

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Discussion

- The condition we provide is easy to be verified algorithmically. In fact, there exist deterministic polynomial-time algorithms, introduced in [Garg et al., FOCS, 2016] and [Ivanyos et al., arXiv, 2016], to determine the existence of shrunk subspaces when the matrix space is given.

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Discussion

- The condition we provide is easy to be verified algorithmically. In fact, there exist deterministic polynomial-time algorithms, introduced in [Garg et al., FOCS, 2016] and [Ivanyos et al., arXiv, 2016], to determine the existence of shrunk subspaces when the matrix space is given.
- This problem is a natural generalization of the Entanglement of Assistance introduced in [D. P. DiVincenzo et al., QCQC, 1999] and [Smolin et al., PRA, 2005]. It also illustrates a connection between the tripartite-to-bipartite entanglement transformation and the structure of matrix spaces based on its singularity.

Thank you!
Questions?