The Theory of Statistical Comparison
with Applications in Quantum Information Science

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these slides are available for download at http://goo.gl/5toR7X

Prerequisites

Prerequisites for the first part (general results):

- ✔ basics of probability and information theory: random variables, joint and conditional probabilities, expectation values, etc
- ✔ in particular, noisy channels as probabilistic maps between two sets $w : \mathscr{A} \to \mathscr{B}$: given input $a \in \mathscr{A}$, the probability to have output $b \in \mathscr{B}$ is given by conditional probability $w(b|a)$
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Prerequisites for the second part (applications):

- ✔ resource theories, in particular, quantum thermodynamics: idea of the general setting and of the problem treated (in particular, some knowledge of majorization theory is helpful)
- ✔ entanglement and quantum nonlocality: general ideas such as Bell inequalities, nonocal games, entangled states, etc
- ✔ open systems dynamics: basic ideas such as reduced dynamics, Markov chains and Markovian evolutions, divisibility, etc (quantum case only sketched, see references)

Part I

Statistical Comparison: General Results

Statistical Games (*aka* Decision Problems)

- ✔ **Definition.** A statistical game is a triple $(\Theta, \mathscr{U}, \ell)$, where $\Theta = \{\theta\}$ and $\mathscr{U} = \{u\}$ are finite sets, and ℓ is a function $\ell(\theta, u) \in \mathbb{R}$.

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- ✔ **Interpretation.** We assume that θ is the value of a parameter influencing what we observe, but that cannot be observed "directly." Now imagine that we have to choose an action u, and that this choice will earn or cost us $\ell(\theta, u)$. For example, θ is a possible medical condition, u is the choice of treatment, and $\ell(\theta, u)$ is the overall "efficacy."

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- ✔ **Resource.** Before choosing our action, we are allowed "to spy" on θ by performing an experiment (i.e., visiting the patient). Mathematically, an experiment is given as a sample set $\mathscr{X} = \{x\}$ (i.e., observable symptoms) together with a conditional probability $w(x|\theta)$ or, equivalently, a family of distributions $\{w_\theta(x)\}_{\theta \in \Theta}$.

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- ✔ **Example in information theory.** Imagine that θ is the input to a noisy channel, x is the output we receive, and u is the message we decode.

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- ✔ **Today’s tutorial.** Basic results of statistical comparison, some quantum generalizations, and finally some applications (quantum thermodynamics, quantum nonlocality, open quantum systems dynamics).

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✔ as a diagram:

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classical case	**quantum case**
• statistical game $(\Theta, \mathscr{U}, \ell)$ • sample set $\mathscr{X}$ • experiment $w = \{w_\theta(x)\}$ • probabilistic decision $d(u\vert x)$ • $p_c(u,\theta) = \sum_x d(u\vert x) w(x\vert\theta) \frac{1}{\vert\Theta\vert}$ • $\mathbb{E}_\ell[w] = \max_{d(u\vert x)} \sum \ell(\theta,u) p_c(u,\theta)$	• statistical game $(\Theta, \mathscr{U}, \ell)$ • Hilbert space $\mathcal{H}_S$ • ensemble $\mathcal{E} = \{\rho_S^\theta\}$ • POVM (measurement) $\{P_S^u\}$ • $p_q(u,\theta) = \mathrm{Tr}\left[\rho_S^\theta \ P_S^u\right] \frac{1}{\vert\Theta\vert}$ • $\mathbb{E}_\ell[\mathcal{E}] = \max_{\{P_S^u\}} \sum \ell(\theta,u) p_q(u,\theta)$
$\Theta \xrightarrow{\text{experiment}} \mathscr{X} \xrightarrow{\text{decision}} \mathscr{U}$ $\theta \longrightarrow x \longrightarrow u$	$\Theta \xrightarrow{\text{ensemble}} \mathcal{H}_S \xrightarrow{\text{POVM}} \mathscr{U}$ $\theta \longrightarrow \rho_S^\theta \longrightarrow u$

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✔ **Remark.** The same statistical game $(\Theta, \mathscr{U}, \ell)$ can be played with classical resources (statistical experiments and decisions) or quantum resources (ensembles and POVMs).

Comparison of Quantum Ensembles (Vanilla Version)

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- ✔ **Side remark.** In fact, the map $\mathcal{L}$ is somewhat more than just positive on $\mathcal{E}_\mathbb{C}$: it is a quantum statistical morphism on $\mathcal{E}_\mathbb{C}$.

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- ✔ **Side remark.** In fact, the map $\mathcal{L}$ is somewhat more than just positive on $\mathcal{E}_\mathbb{C}$: it is a quantum statistical morphism on $\mathcal{E}_\mathbb{C}$. In general:

$$\text{PTP on } \mathsf{L}(\mathcal{H}_S) \underset{\not\Leftarrow}{\Longrightarrow} \text{stat. morph. on } \mathcal{E}_\mathbb{C} \underset{\not\Leftarrow}{\Longrightarrow} \text{PTP on } \mathcal{E}_\mathbb{C}$$

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- ✔ as a diagram:

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\begin{array}{ccccccc}
\Theta & \xrightarrow{\text{ensemble}} & \mathcal{H}_S & \xrightarrow{\text{quantum noise}} & \mathcal{H}_{S'} & \xrightarrow{\text{POVM}} & \mathscr{U} \\
\wr\!\downarrow & & \wr\!\downarrow & & \wr\!\downarrow & & \wr\!\downarrow \\
\theta & \xrightarrow[\mathcal{E}]{} & \rho_S^\theta & \xrightarrow[\Phi]{} & \sigma_{S'}^\theta & \xrightarrow[\{Q_{S'}^u\}]{} & u
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Compositions of Ensembles

- ✔ consider two parameter sets, $\Theta = \{\theta\}$ and $\Omega = \{\omega\}$, two Hilbert spaces, $\mathcal{H}_S$ and $\mathcal{H}_R$, and two ensembles, $\mathcal{E} = \{\rho_S^\theta\}_{\theta\in\Theta}$ and $\mathcal{F} = \{\tau_R^\omega\}_{\omega\in\Omega}$. Then we denote as $\mathcal{F} \otimes \mathcal{E}$ the ensemble $\{\tau_R^\omega \otimes \rho_S^\theta\}_{\omega\in\Omega,\theta\in\Theta}$

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- ✔ **Extended comparison.** Given two ensembles $\mathcal{E} = \{\rho_S^\theta\}$ and $\mathcal{E}' = \{\sigma_{S'}^\theta\}$, we can supplement them both with the same extra ensemble $\mathcal{F} = \{\tau_R^\omega\}$ and play statistical games $(\Omega \times \Theta, \mathscr{U}, \ell)$.

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Part II

Applications to Quantum Information Science

Section 1

Quantum Thermodynamics

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Given two dichotomies $w = \{w_1(x), w_2(x)\}$ *and* $w' = \{w'_1(y), w'_2(y)\}$, $w \succ w'$ *if and only if* $\mathbb{E}_\ell[w] \geqslant \mathbb{E}_\ell[w']$ *for all binary statistical games.*

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- ✔ in formula: for classical dichotomies, $w \succ w' \iff w \succ_2 w'$. (The symbol $\succ_2$ denotes the information ordering restricted to binary statistical games.)

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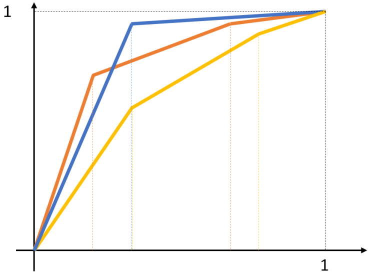

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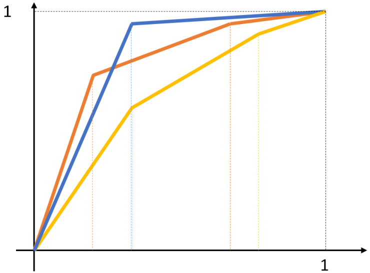

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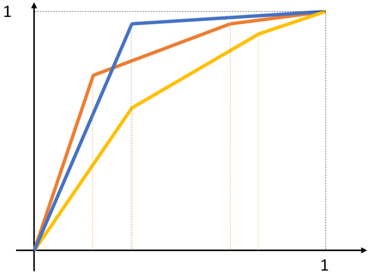

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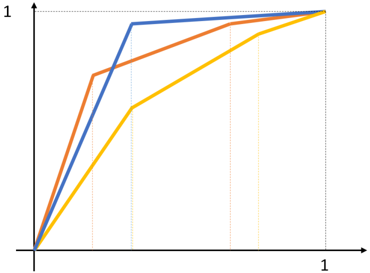

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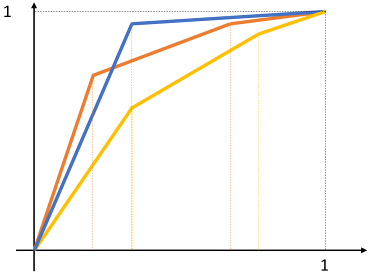

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Given a binary ensemble $\mathcal{E} = \{\rho_1, \rho_2\}$, define the curve $L(\rho_1, \rho_2)$ as the upper boundary of the region $\mathcal{R}(\rho_1, \rho_2) \triangleq \{(x, y) = (\mathrm{Tr}[E\ \rho_2], \mathrm{Tr}[E\ \rho_1]) : 0 \leqslant E \leqslant \mathbb{1}\}$

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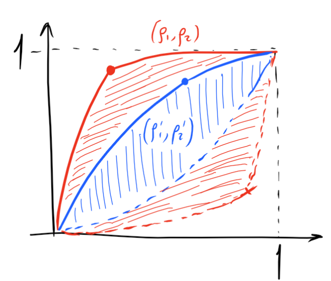

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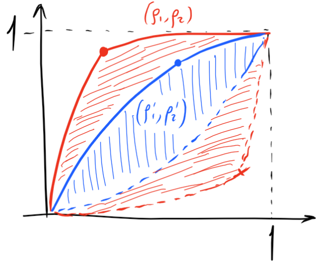

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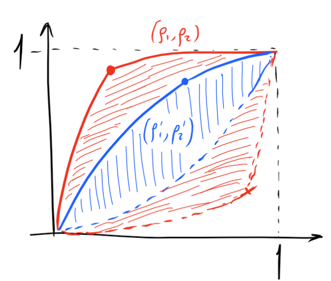

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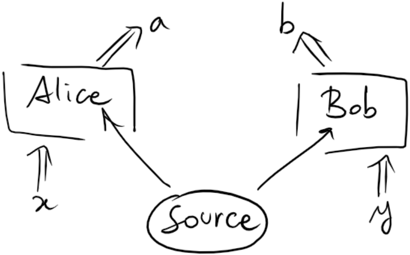
- ✔ **Fact 2.** A result by Alberti and Uhlmann (1980) implies that, if both quantum ensembles are on $\mathbb{C}^2$, then $L(\rho_1, \rho_2) \geqslant L(\rho_1', \rho_2')$ if and only if there exists a CPTP map Φ such that $\Phi(\rho_i) = \rho_i'$ for $i = 1, 2$

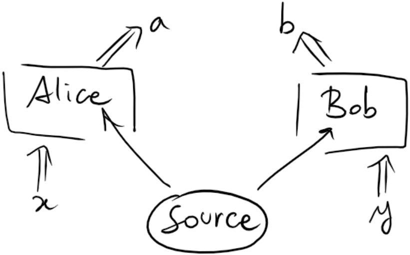
Section 2

Entanglement and Quantum Nonlocality

Nonlocal Games



- ✔ a nonlocal game (Bell inequality) is a bipartite decision problem played "in parallel" by space-like separated players; it is formally given as $G = (\mathscr{X}, \mathscr{Y}; \mathscr{A}, \mathscr{B}; \ell)$



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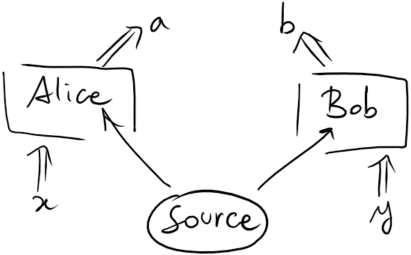

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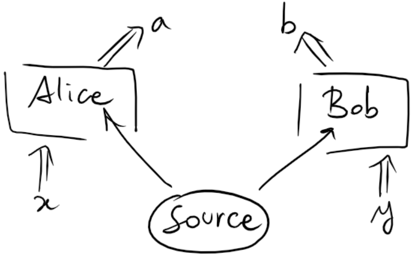

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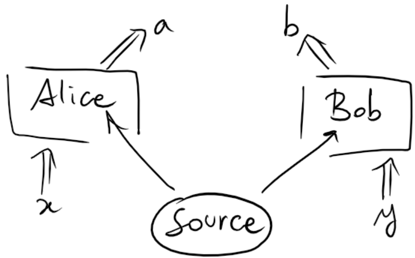
Nonlocal Games



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Comparison of Bipartite Quantum States

- ✔ consider two bipartite density operators: ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma_{A'B'}$ on $\mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$

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We say that ρ_{AB} is more nonlocal than $\sigma_{A'B'}$, in formula, $\rho_{AB} \succ_{nl} \sigma_{A'B'}$, if and only if $\mathbb{E}_G[\rho_{AB}] \geqslant \mathbb{E}_G[\sigma_{A'B'}]$ for all nonlocal games $G = (\mathscr{X}, \mathscr{Y}; \mathscr{A}, \mathscr{B}; \ell)$.

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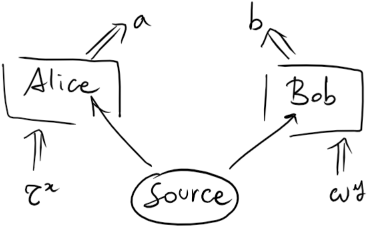
- ✔ **Hidden Nonlocality.** Werner (1989) showed that there exist entangled bipartite states that do not exceed the classical value, for all possible Bell inequalities

Quantum Nonlocal Games



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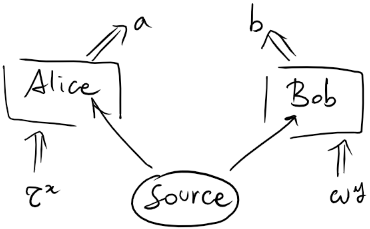

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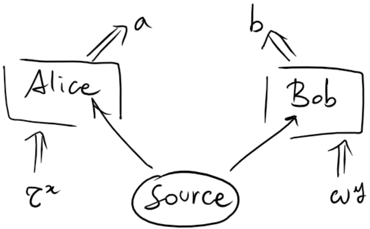

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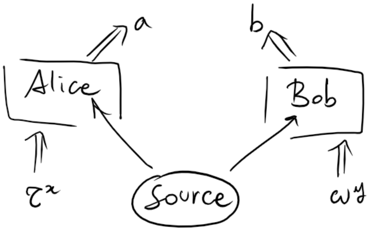

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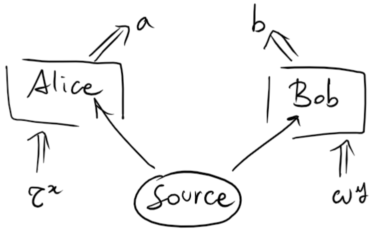

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- application: measurement-device independent entanglement witnesses (MDIEW)

Section 3

Open Systems Dynamics

✔ recall: a statistical game is as follows

$$\begin{array}{ccccc} \Theta & \xrightarrow{\text{experiment}} & \mathscr{X} & \xrightarrow{\text{decision}} & \mathscr{U} \\ \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\ \theta & \xrightarrow[w(x|\theta)]{} & x & \xrightarrow[d(u|x)]{} & u \end{array} \Longrightarrow \ell(\theta, u)$$

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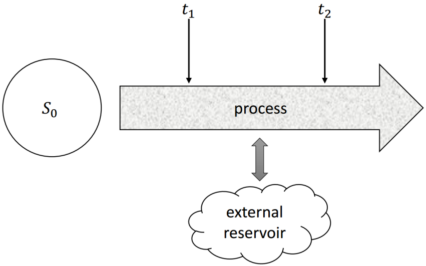
- a communication game is a triple $(\mathscr{U}, \Theta, e(\theta|u))$ and the payoff is the probability of guessing the message correctly:

$$P^e_{\text{guess}}[w] \triangleq \max_{d(\hat{u}|x)} \sum_{u,\theta,x} d(u|x) w(x|\theta) e(\theta|u) \frac{1}{|\mathscr{U}|}$$

Divisible Evolutions



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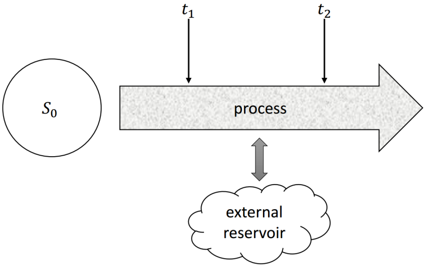

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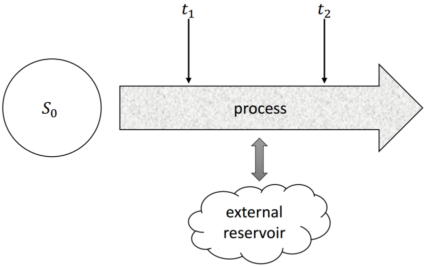

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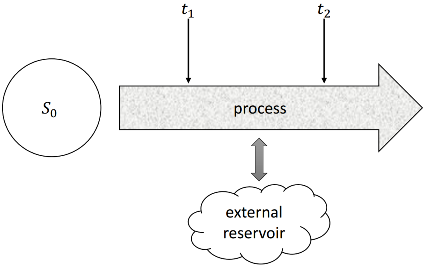

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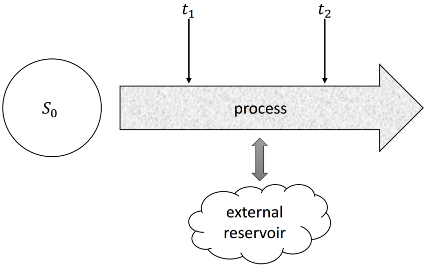

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Theorem (Blackwell's Theorem for Open Systems Dynamics)

The evolution $t_0 \to t_1 \to t_2$ is divisible if and only if $P^e_{\text{guess}}[w_1] \geqslant P^e_{\text{guess}}[w_2]$ for all communication games $(\mathscr{U}, \Theta, e(\theta|u))$

- ✔ for the quantum case, see the references for further details

Essential Bibliography

(this list is not meant to be an exhaustive bibliography, but only a selection of accessible, introductory, mostly self-contained texts on the topics covered in this lecture)

General theory:

- ✔ D. Blackwell and M.A. Girshick, *Theory of games and statistical decisions*. (Dover Publications, 1979).
- ✔ A.S. Holevo, *Statistical decision theory for quantum systems*. Journal of Multivariate Analysis **3**, 337–394 (1973).
- ✔ P.K. Goel and J. Ginebra, *When is one experiment 'always better than' another?* Journal of the Royal Statistical Society, Series D (The Statistician) **52**(4), 515–537 (2003).
- ✔ F. Liese and K.-J. Miescke, *Statistical decision theory*. (Springer, 2008).
- ✔ F. Buscemi, *Comparison of quantum statistical models: equivalent conditions for sufficiency*. Communications in Mathematical Physics **310**(3), 625–647 (2012). arXiv:1004.3794 [quant-ph].

Quantum Lorenz curves:

- ✔ J.M. Renes, *Relative submajorization and its use in quantum resource theories*. arXiv:1510.03695 [quant-ph].
- ✔ F. Buscemi and G. Gour, *Quantum relative Lorenz curves*. arXiv:1607.05735 [quant-ph].

Quantum nonlocal games:

- ✔ F. Buscemi, *All entangled states are nonlocal.* Physical Review Letters **108**, 200401 (2012).

Open quantum systems dynamics:

- ✔ F. Petruccione and H.-P. Breuer, *The Theory of Open Quantum Systems*. (Oxford University Press, Oxford, 2002).
- ✔ A. Rivas, S.F. Huelga, and M. B. Plenio, *Quantum non-Markovianity: characterization, quantification and detection*. Reports on Progress in Physics **77**, 094001 (2014).
- ✔ F. Buscemi and N. Datta, *Equivalence between divisibility and monotonic decrease of information in classical and quantum stochastic processes*. Physical Review A **93**, 012101 (2016).
- ✔ F. Buscemi, *Reverse data-processing theorems and computational second laws*. arXiv:1607.08335 [quant-ph].

Thank You

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