Quantum error-correction in black holes

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[1] Holographic quantum error-correcting codes,

[2] Chaos in quantum channel,


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A black hole is a quantum error-correcting code
Anti-de Sitter/Conformal field theory (AdS/CFT) correspondence

• [Conjecture] Equivalence of string (gravity) theory in bulk with CFT on boundary (Maldacena)
Anti-de Sitter/Conformal field theory (AdS/CFT) correspondence

- [Conjecture] Equivalence of string (gravity) theory in bulk with CFT on boundary (Maldacena)

- [Holography] Bulk degrees of freedom are encoded in boundary, like a hologram.

Hyperbolic space (negatively curved)
Quantum entanglement in AdS/CFT

- [Ryu-Takayanagi formula]

\[ S(A) = \frac{1}{4G_N} \min_{\gamma_A} \text{area}(\gamma_A) \]

Minimize over spatial bulk surfaces homologous to A

(For "large N")
Space-time as a tensor network

• [Swingle’s conjecture]

AdS/CFT correspondence can be expressed as a MERA?

\[
S(A) \leq \frac{1}{4G_N} \min_{\gamma_A}(\text{area}(\gamma_A))
\]

MERA = Multi-scale Entanglement Renormalization Ansatz (Vidal)
Concrete and simple toy tensor network model

(joint with Harlow, Pastawski and Preskill)
Dictionary : Correspondence of operators

boundary system

bulk system

\( O \)

\( O_1 \)

\( O_2 \)

\( \phi \)

\( \phi_1 \)

\( \phi_2 \)
• [Entanglement wedge reconstruction]

A bulk operator $\phi$ can be represented by some integral of local boundary operators supported on $A$ if $\phi$ is contained inside the entanglement wedge of $A$.

Remarks

• Entanglement wedge may go beyond black hole horizons (i.e. no firewall).

• “Proven” by using a generalized RT formula (Jefferis et al, Dong et al, Bao et al)

• No explicit recipe is known for more than one intervals
Bulk locality puzzle

• The reconstruction recipe leads to a paradox

All the bulk operators must correspond to identity operators on the boundary?

If so, the AdS/CFT seems very boring …
Quantum error-correction in AdS/CFT?

- The AdS/CFT correspondence can be viewed as a quantum error-correcting code [Almheiri-Dong-Harlow].

They are different operators, but act in the same manner in a low energy subspace.

cf. Quantum secret-sharing code
Let’s construct a toy model
A simple toy model

1 bulk qubit
5 boundary qubits

in total, just 6 qubits

A bulk operator must have representations on any region with three qubits.

Entanglement wedge reconstruction!
Five-qubit code

- Encode a single logical qubit into a system of five qubits.
  \[
  S_1 = X \otimes Z \otimes Z \otimes X \otimes I \\
  S_2 = I \otimes X \otimes Z \otimes Z \otimes X \\
  S_3 = X \otimes I \otimes X \otimes Z \otimes Z \\
  S_4 = Z \otimes X \otimes I \otimes X \otimes Z 
  \]
- Codeword space
  \[
  C = \{ |\psi\rangle : S_j |\psi\rangle = |\psi\rangle \ \forall j \}
  \]
- Five-qubit code has code distance 3

The code can correct single-qubit errors!
Five-qubit code is a quantum gravity
Perfectness of five qubit code

- Let’s view the five-qubit code as a six-leg tensor.

- Any leg can be used as an input of quantum codes.
A holographic quantum error-correcting code

- A tiling of the five qubit code
Entanglement wedge reconstruction

- 1 in & 3 out (operator pushing)
Holographic state

• The Ryu-Takayanagi formula holds exactly (tiling of perfect tensors)

Coarse-graining (RG transformation) = Distillation of EPR pairs along the geodesic
Perfect tensors

- A pure state with maximal entanglement in any bipartition

\[ \rho_A \propto I_A \] for all \( |A| \leq n \)

- Five qubit code

\[ |\psi\rangle = \sum_{i_1=0}^{v-1} \sum_{i_2=0}^{v-1} \cdots \sum_{i_n=0}^{v-1} T_{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle \]

- Perfect tensor (2n legs)

6 leg perfect tensor
Random tensors

• Perfect tensors are very rare…

• But almost perfect tensors are pretty common!

Pick a Haar random state.

Due to the Page’s theorem, the state is almost maximally entangled along any cut.

• To construct a holographic code/state, just pick tensors randomly.
Coding properties: erasure threshold

- Remove qubits with probability $p = \frac{1}{2} - \epsilon$

A central bulk leg is contained in the entanglement wedge of $A$ (if $|A|>|B|$)

Erasure threshold = $1/2$

New quantum codes from quantum gravity?
So far, no black holes...
Information loss puzzle

- Is quantum information lost?

\[ |\psi\rangle \]

Hawking radiation

\[ e^{-\beta H} \]
Information loss puzzle

• Or hidden into some non-local degrees of freedom?

Locally it looks like $e^{-\beta H}$, but globally it is not.
Quantum error-correction

- Scrambling is very similar to how quantum error-correcting codes work.

- Local indistinguishability.

\[ \rho_R \approx \rho'_R \]

\[ e^{-iH t} \]

logical qubit

black hole

Hawking radiation = error

codewords
Choi-Jamilkowski isomorphism

- Quantum channel on n qubits can be viewed as a state on 2n qubits.

unitary operator as a state

\[ U = \sum_{i,j} U_{i,j} |i\rangle \langle j| \quad |U\rangle = \sum_{i,j} U_{i,j} |i\rangle \otimes |j\rangle \]

\[ \rho_{in} = \{ p_j, |\psi_j\rangle \} \quad \rho_{out} = \{ p_j, |\phi_j\rangle \} \]

\[ |\Psi\rangle = \sum_j \sqrt{p_j} |\psi_j\rangle \otimes |\phi_j\rangle \]

[Choi-Jamilkowski, Hayden-Preskill, Hartman-Maldacena]
Scrambling in a black hole

• Thermofield double state (finite T)

\[
\rho_{in} = \frac{e^{-\beta H}}{\text{Tr} \ e^{-\beta H}} \quad \rho_{out} = \frac{e^{-\beta H}}{\text{Tr} \ e^{-\beta H}}
\]

\[
|\Psi\rangle = \sum_j e^{-\beta E_j/2} e^{-iE_j t} \psi_j \otimes |\psi_j\rangle
\]

• A black hole geometry for the TFD state is the two-sided hole (AdS/CFT prediction)

Let's construct a tensor network toy model!
Toy model of the Einstein-Rosen bridge

- Consider a network of random unitary operators, tiling the wormhole geometry.
How do we probe the interior of a black hole?
Out-of-time ordered correlation functions

• We should measure some “hidden” correlations (Kitaev 2014)

\[ OTO = \langle A(0)B(t)[C(0)D(t)] \rangle \]

local operators

\[ B(t) = e^{-iHt}B(0)e^{iHt} \]
\[ D(t) = e^{-iHt}D(0)e^{iHt} \]

• Previously considered by Larkin and Ovchinikov in 1960s, and recently by Shenker and Stanford
Time evolution of operators

• OTOCs detect the growth of operators

• Consider

\[ OTO = \langle A(0)B(t)A^\dagger(0)B^\dagger(t) \rangle \]

\[ [A, B] = 0 \text{ then } OTO = 1 \]

\[ \{A, B\} = 0 \text{ then } OTO = -1 \]

• Expand B(t):

\[ B(t) = e^{-iHt}Be^{iHt} = \sum_j \alpha_j P_j \]

Non-commutativity between A(0) and B(t)

[Roberts-Stanford-Susskind]
Key Questions

• How do we define scrambling?

• Quantum information theoretic meaning of OTO?

• Is the converse true?

I will relate OTOCs to entanglement entropies (joint with Hosur, Qi and Roberts)
[Theorem] **Average** value of OTO

- Average of OTO over local operators A and D at $T=\infty$

$$\left| \left\langle A(0) D(t) A^\dagger(0) D^\dagger(t) \right\rangle \right|$$

**average** over A, D

[Hosur-Qi-Roberts-BY]
[Theorem] **Average value of OTO**

- Average of OTO over local operators A and D at T=\(\infty\)

\[
\left| \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle \right| = \frac{1}{4^{a+d}} \sum_{A,D} \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle
\]

average over A, D

Pauli operators (unitary 1-design)
[Theorem] **Average value of OTO**

- Average of OTO over local operators $A$ and $D$ at $T=\infty$

\[
\left| \left\langle A(0) D(t) A^\dagger(0) D^\dagger(t) \right\rangle \right| = \frac{1}{4^{a+d}} \sum_{A,D} \left\langle A(0) D(t) A^\dagger(0) D^\dagger(t) \right\rangle
\]

average over $A$, $D$

- Pauli operators (unitary 1-design)

![Diagram of quantum circuit](image_url)

[Hosur-Qi-Roberts-BY]
[Theorem] **Average value of OTO**

- Average of OTO over local operators A and D at $T=\infty$

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\left| \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle \right| = \frac{1}{4^{a+d}} \sum_{A,D} \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle
\]

Average over A, D

Pauli operators (unitary 1-design)

\[= 2^{n-a-d-S_{BD}^{(2)}} \]

Renyi-2 entropy

[Hosur-Qi-Roberts-BY]
[Theorem] **Average value of OTO**

- Average of OTO over local operators $A$ and $D$ at $T=\infty$

$$\left| \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle \right| = \frac{1}{4^{a+d}} \sum_{A,D} \langle A(0) D(t) A^\dagger(0) D^\dagger(t) \rangle$$

- Average over $A$, $D$

- Pauli operators (unitary 1-design)

- Renyi-2 entropy $S_{BD}^{(2)}$

- If $\text{OTO} \approx 0$ then, $S_{BD}^{(2)}$ is large

This implies the **mutual information** $I_{BD}^{(2)} = S_B^{(2)} + S_D^{(2)} - S_{BD}^{(2)}$ is small

- $B$ and $D$ are not correlated, so the system is **scrambling**.

[Hosur-Qi-Roberts-BY]
[Theorem] **Average value of OTO**

- Average of OTO over local operators $A$ and $D$ at $T=\infty$
  
  $$\left| \langle A(0)D(t)A^\dagger(0)D^\dagger(t) \rangle \right| = \frac{1}{4^{a+d}} \sum_{A,D} \langle A(0)D(t)A^\dagger(0)D^\dagger(t) \rangle$$

  - Average over $A$, $D$
  - Pauli operators (unitary 1-design)
  - Renyi-2 entropy

- If $OTO \simeq 0$, then, $S_{BD}^{(2)}$ is large
  
  This implies the mutual information $I_{BD}^{(2)} = S_B^{(2)} + S_D^{(2)} - S_{BD}^{(2)}$ is small

  - $B$ and $D$ are not correlated, so the system is **scrambling**.

- For finite $T$, we will consider the so-called **Thermofield double state**.

[Hosur-Qi-Roberts-BY]
Scrambling phenomena in a black hole

• This captures key properties of **scrambling** for “large-N” theories.
  
  Eg) Ballistic propagations of entanglement, RT formula in a wormhole geometry…

• Operator size grows **linearly**, OTO will pick it up.

• For a **non-local** random quantum circuit, the scrambling time is $\log(n)$ [Cleve et al 2006]

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**Figure 10:** Tensor network representation of the Einstein-Rosen bridge. We will consider a network of perfect tensors.

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**Random unitary (maximally entangled)**
Scrambling in AdS black hole

mutual information $I(A, B)$ is large

mutual information $I(A, B)$ is (almost) zero

OTO correlators will be small.
What did we learn?
Lesson 1

The AdS/CFT correspondence is a quantum error-correcting code.

Bulk quantum information is encoded in boundary like a hologram.

very entangled tensor (eg random tensor)
Lesson 2

OTO correlator is the probe of space-time

Before we begin, let us review the proposal of [12]. The tensor network representation of the thermofield double state is shown in Fig. 10. At the left and right ends, we have a hyperbolic network, representing the two asymptotically AdS boundaries. This network extends infinitely from the UV into the IR thermal scale at the black hole horizon. Then, the middle is flat representing the black hole interior. The entire network grows as $t$ grows by adding more layers in the middle flat region.

We would like to further elaborate on this proposal of tensor network representation of the black hole interior. We will study networks of perfect tensors and demonstrate chaotic dynamics by finding ballistic growth of local unitary operators and the linear growth of the tripartite information until the scrambling time. For the rest of discussion, we take the infinite temperature limit so we can ignore the hyperbolic part and focus in on the planar tiling of tensor networks representing the interior.

Figure 8: Tensor network representation of the Einstein-Rosen bridge. We will consider a network of perfect tensors.

5.1 Ballistic propagation of entanglement

Let us begin by briefly reviewing a definition of perfect tensors. Consider a tensor $T$ with $2^n$ legs and bond dimension $v$. A tensor can be represented as a pure state $|i\rangle = X_{i_1},...,i_{2^n} T_{i_1},...,i_{2^n} |i_1\rangle,...,i_{2^n}\rangle$ with a proper normalization. We call a tensor perfect if it is associated with a pure state $|i\rangle$, called a perfect state, which is maximally entangled along any bipartition. Namely, $S_A = |A| \cdot \log_v |A| s.t |A| = n.$

OTO correlator detects scrambling/chaos
OTO correlators are the probes of space-time (seeing the interior of a black hole)

\[ \text{OTO} = \langle A(0)B(t)C(0)D(t) \rangle \]
1970s

Black hole =

2010s

Black hole =
[1] Holographic quantum error-correcting codes

[2] Chaos in quantum channel,


Thank you!