Fault-tolerant error correction for non-abelian anyons

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AQIS’16
September 2, 2016

1arXiv:1607.02159
Outline

1. Non-abelian anyons and quantum information
2. Error correction for abelian anyons
3. Error correction for non-abelian anyons
What are anyons\textsuperscript{1}? 

- Localized gapped excitations living on a 2-dimensional surface

\textsuperscript{1}A. Kitaev, Annals Phys. \textbf{321}, 2-111 (2006)
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- We can imagine bringing 2 excitation together (\(a\) and \(b\)), and ask what is their total charge \(c\).
- The possible outcomes are given by the fusion rules:

\[
    a \times b = \sum_c N_{ab}^c c
\]

Abelian vs non-abelian anyons

- The fusion rules for abelian anyons are deterministic and unique, as for excitations in the toric code:
  \[ e \times e = 1, \ m \times m = 1, \ e \times m = em, \ldots \]

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- A Hilbert space is associated to each fusion/splitting process.

- Fusing two anyons \( a_1 \) and \( a_2 \) collapses the wavefunction into a definite super-selection sector, with probability given by Born’s rule:
  \[
P(c) = \langle \psi | \Pi_c^{a_1 a_2} | \psi \rangle.
\]  \( (1) \)
Quantum computation with non-abelian anyons

Measurement

Applying gates

Initialization

Thermal processes can corrupt the information\textsuperscript{1}

- At $T > 0$, thermal excitations are present in finite density.
- Thermal excitations can diffuse at no energy cost.
- It really is a scalability issue: for large systems, such processes are bound to happen.

\textsuperscript{1}F. L. Pedrocchi et al., arXiv:1505.03712
Our goal is to find an error correction procedure for systems of non-abelian anyons.

\(^1\)Dennis et al., J. Math. Phys. 43, 4452 (2002)
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We want to include measurement errors
Our goal is to find an error correction procedure for systems of non-abelian anyons. We want to include measurement errors. Fault-tolerant error correction for topologically ordered systems giving rise to abelian anyons have been studied extensively.¹

Anyons and topological order

Anyons appear as excitations in topologically ordered systems\(^1\). The ground space is degenerate and quantum information can be encoded in such states.

- Logical operations consist of creating a pair of excitations, performing non-trivial loop, and fuse the excitations back to the vacuum.
- World lines with the same topology have the same effect on the ground space.

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- A decoding algorithm is used to find a correction procedure.
- The correction operations are performed.
Various families of decoding algorithms

- Perfect matching
- Mapping to statistical physics problems
- Clustering methods
- Cellular automaton
- Renormalization methods

\[a\] G. Duclos-Cianci et al., PRL 104, 050504 (2010)
Emerging structure of the noise

- Each actual error is characterized by a level $n$.
- If fits in a box of size $Q^n \times Q^n \times U^n$ and is separated by at least $aQ^n$ sites ($bU^n$ time steps) from other actual errors.
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The notion of actual error is recursively defined over the level.

The rate of appearance of a level-$n$ actual errors goes as $\epsilon_n \sim e^{-2^n}$.

The idea behind Harrington’s algorithm

- Cellular automata periodically measure topological charges.

![Diagram of Harrington’s algorithm](image)
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- If 2 excitations are close, they will be fused together.
- If an excitation is isolated, it is displaced to the colony center.
The need for renormalization

- An error chain extending over 2 or more colonies cannot get corrected using such simple local rules.
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- Colonies are periodically grouped into renormalized colonies.
- Renormalized transition rules are periodically applied.
Existence of a threshold

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- A level-\(n\) actual errors gets corrected by the \(n^{th}\) level transition rules.
- Actual errors stay well-separated from each other in time.

The properties above combined with the fact that \(\epsilon_n \sim e^{-2^n}\) leads to the existence of a threshold.
Complications for non-abelian anyons: probabilistic evolution

The fusion channel of 2 or more anyons is in general not deterministic:

\[ \alpha + \beta \]

We introduce the notion of a *trajectory domain* of an error. It roughly corresponds to the set of sites having a probability of becoming charged because of a given error.
Complications for non-abelian anyons: renormalized charge

The total charge present in a colony becomes path-dependent and subject to rapid fluctuations.

The notion of renormalized charge needs to be carefully defined, and must include the interactions of the errors with the transition rules.
Complications for non-abelian anyons: interactions between renormalization levels

The hierarchic classification of errors does not capture the 'topological interaction' between anyons caused by different actual errors.

We introduce the notion of *causally-linked clusters* of errors, sets of actual errors which can potentially interact with each others through the application of transition rules.
Key properties for non-cyclic anyons

Despite all the complications related to the 'non-abelianity', we show that our algorithm is such that
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- Renormalized transition rules are always successful after being applied a constant number of times.

Non-cyclic anyons are anyons such that for any sequence of labels $\{x_0, x_1, \ldots, x_n\}$ such that $x_0 = x_n$ (and not the vacuum), then

$$\prod_{i=0}^{n} N_{x_i x_i}^{x_i+1} = 0.$$
A threshold for non-cyclic anyons

Despite the new failing mechanisms for non-abelian anyons, we show that our algorithm possess a threshold for non-cyclic anyons.
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Threshold theorem

If $A$ is non-cyclic, there exists a critical value $p_c > 0$ such that if $p + q < p_c$, for any number of time steps $T$ and any $\epsilon > 0$, there exists a linear system size $L = Q^n \in \mathcal{O}(\log \frac{1}{\epsilon})$ such that with probability of at least $1 - \epsilon$, the encoded quantum state can in principle be recovered after $T$ time steps.
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The theorem provides an upper bound on the numerical value of $p_c < 2, 7 \times 10^{-20} \times (3D + 1)^{-4}$. 
Numerical simulations

We performed numerical simulations for Ising anyons. They suggest a threshold in the range of $10^{-4} \sim 10^{-3}$. 

![Graph showing average memory lifetime vs error rate](image-url)
Future directions

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- How do we modify the algorithm to the case where we have computational anyons?
- How about braiding in a fault-tolerant manner?
Thank you for your attention!