Maximum privacy without coherence, zero-error

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Zero-error Information, Operators, and Graphs

Taipei, August 29, 2016

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Result:

$\exists$ family of channels N_d w/ input dim d^2, quantum output dim d, and with $P_0(N_d) = \log d$ and $Q_0(N_d) = 0$.

Zero-error private capacity

Zero-error quantum capacity

Result:

$\exists$ family of channels N_d w/ input dim d^2, quantum output dim d,

and with $P_0(N_d) = \log d$ and $Q_0(N_d) = 0$.

Zero-error private capacity

Zero-error quantum capacity

log d max given the quantum output dim

can't be less 0

max difference

Prior work (L, Li, Smith, Smolin 2014):

$\exists$ family of channels N_d w/ input dim d^2, quantum output dim d,

and with $P(N_d) = \log d$ and $Q(N_d) < 1$.

regular
private capacity

transmission
happens to be ZE

regular
quantum capacity

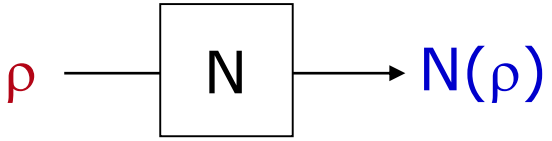
lower bounded
by ≈ 0.6

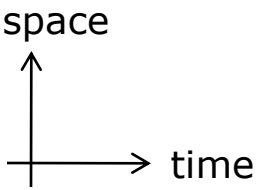
Our contribution: showing $Q_0(N_d) = 0$.

Noisy quantum channels

input from
sender Alice

output to
receiver Bob





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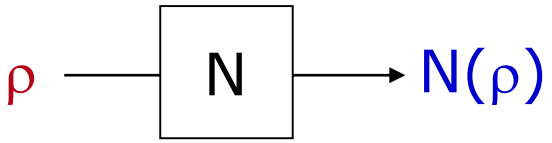

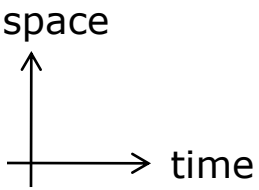

Linear, trace-preserving, completely positive

Most general evolution (as long as the quantum system has no prior correlation with the environment)

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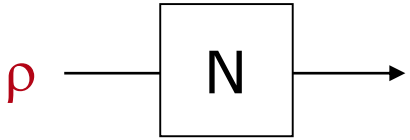

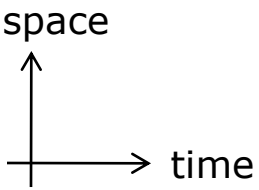

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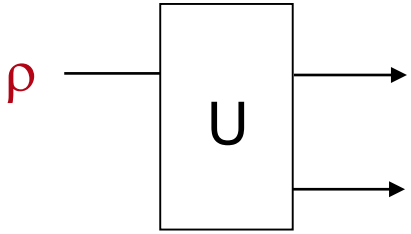
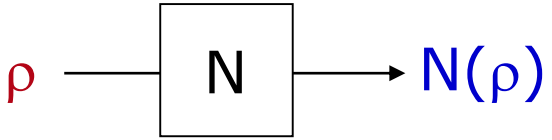


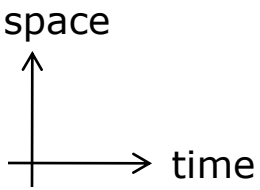
leakage to
environment Eve

Noisy quantum channels

input from sender Alice

output to receiver Bob





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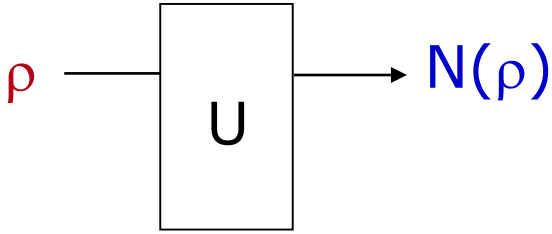

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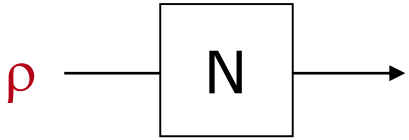

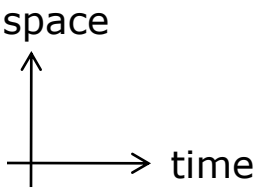

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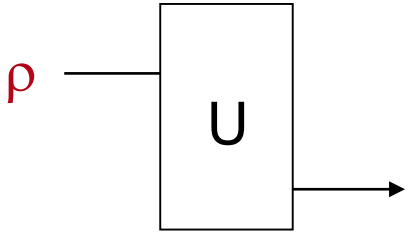

leakage to
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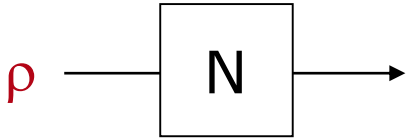

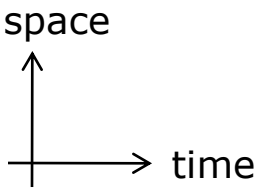

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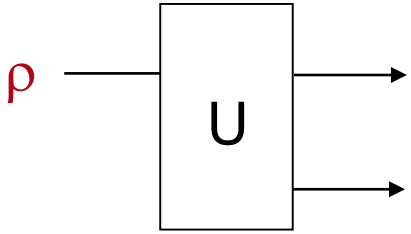
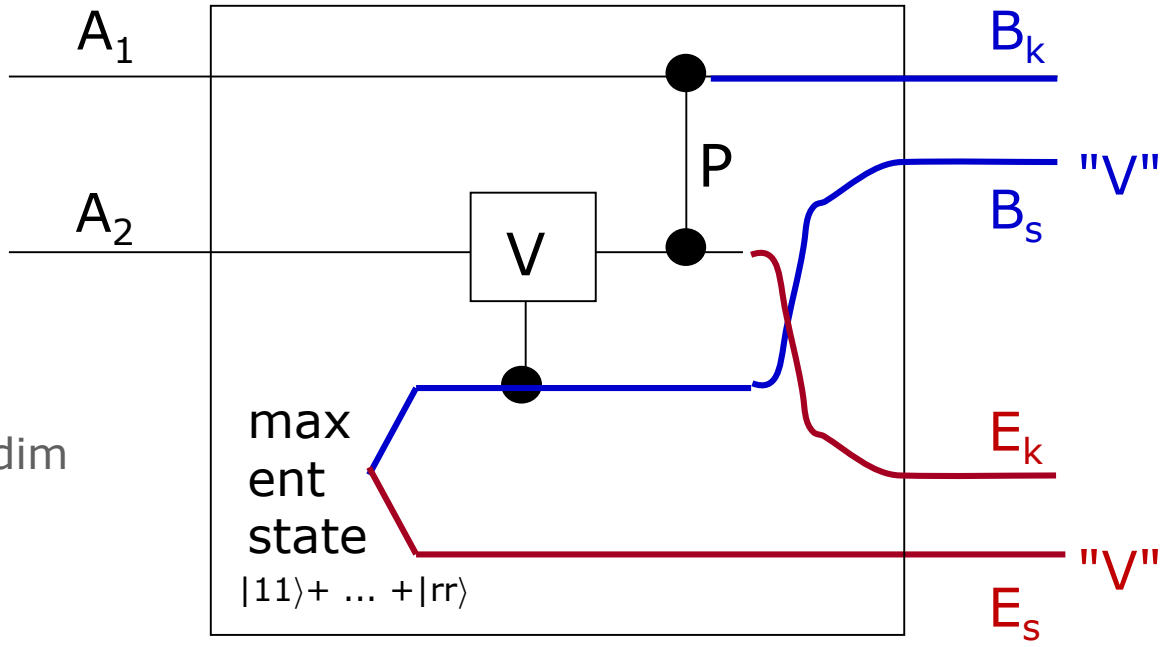


leakage to environment Eve

The channel N_d:



$A_{1,2}$: d-dim
$d_{in} = d^2$

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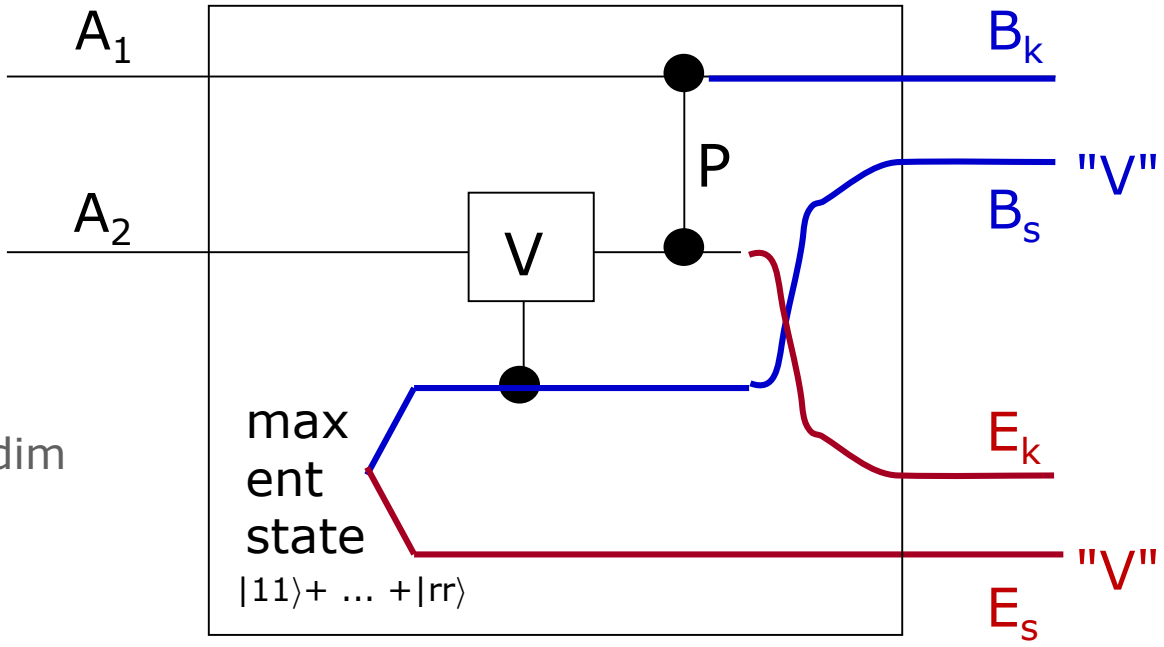
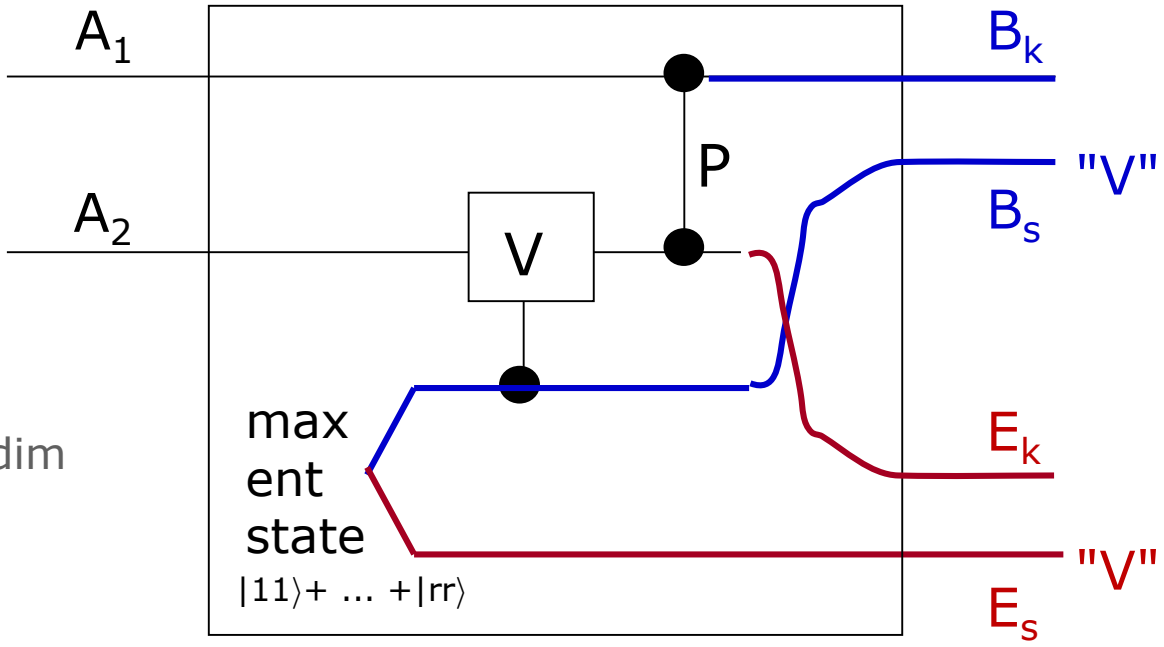


$A_{1,2}$: d-dim
$d_{in} = d^2$

V: random element of a unitary 2-design (e.g., Clifford group).

The channel N_d:



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$P|ij\rangle = \omega^{ij}|ij\rangle$, ω primitive d^{th} root of unity

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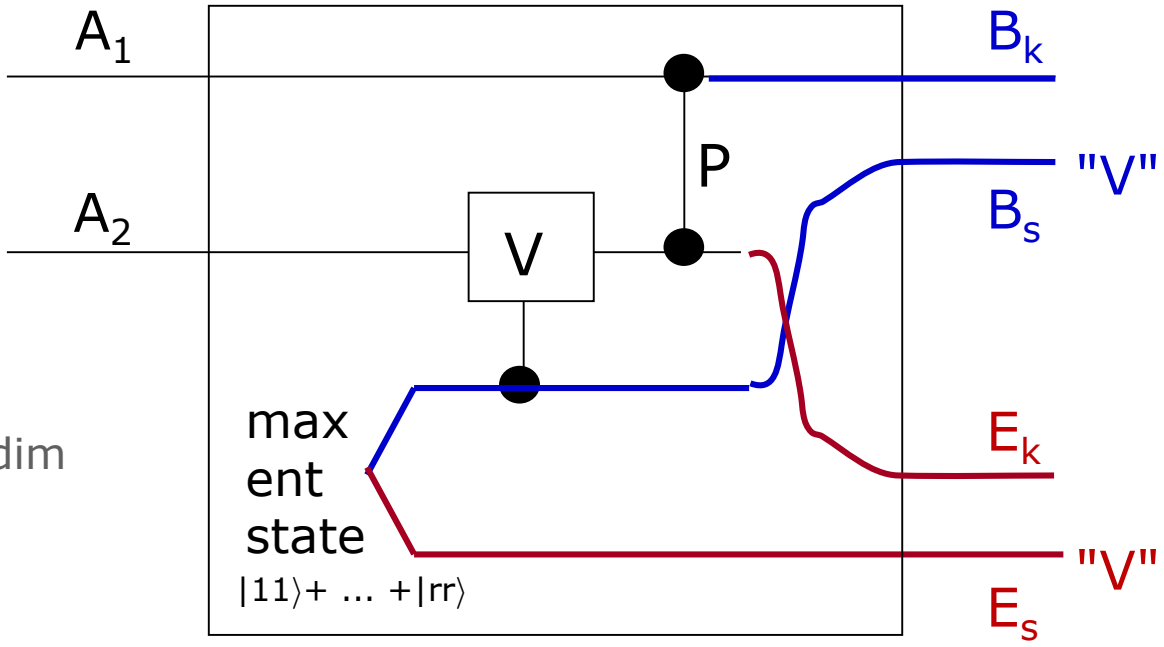

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If A_1 is $|i\rangle$, apply Z^i to A_2 , $Z = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^{d-1} \end{pmatrix}$

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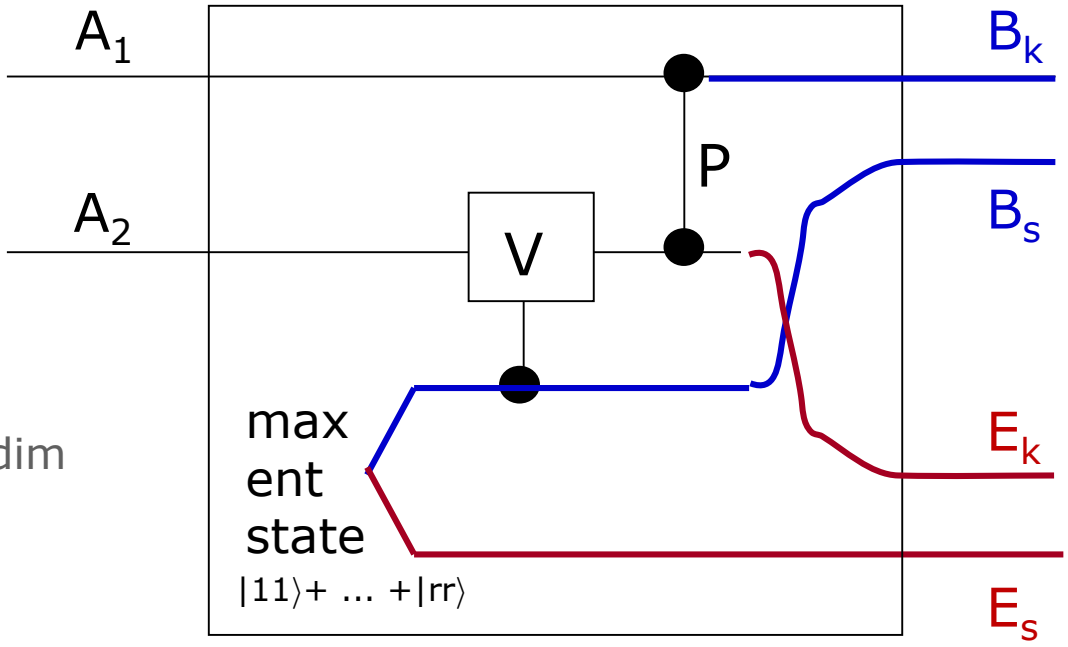

N takes A_1A_2 to B_kB_s

N^c takes A_1A_2 to E_kE_s

$A_{1,2}$: d-dim
$d_{in} = d^2$

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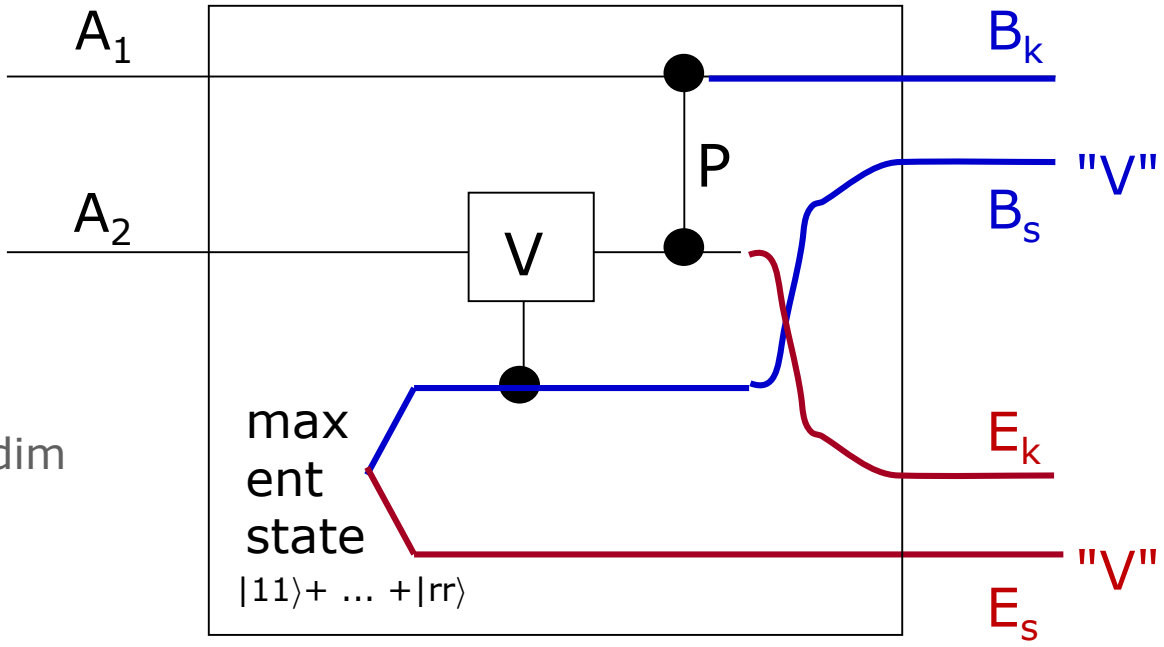
Why $P_0(N_d) = \log d$:



$A_{1,2}$: d-dim
$d_{in} = d^2$

The Choi state is an exact p-bit (HHHO03) with log d key-bits. (Note achievable in 1-shot.)

Why it's not obvious $Q_0(N_d) = 0$:

- The channel N_d together with classical feedback or with entanglement has a high zero-error quantum capacity

- Superactivation in zero-error capacities of a quantum channel

  Duan 2009, Chen-Cubitt-Harrow 2011, Cubitt-Smith 2012
  Shirokov-Shulman-2014, Shirokov-Shulman-2015, Shirokov 2015

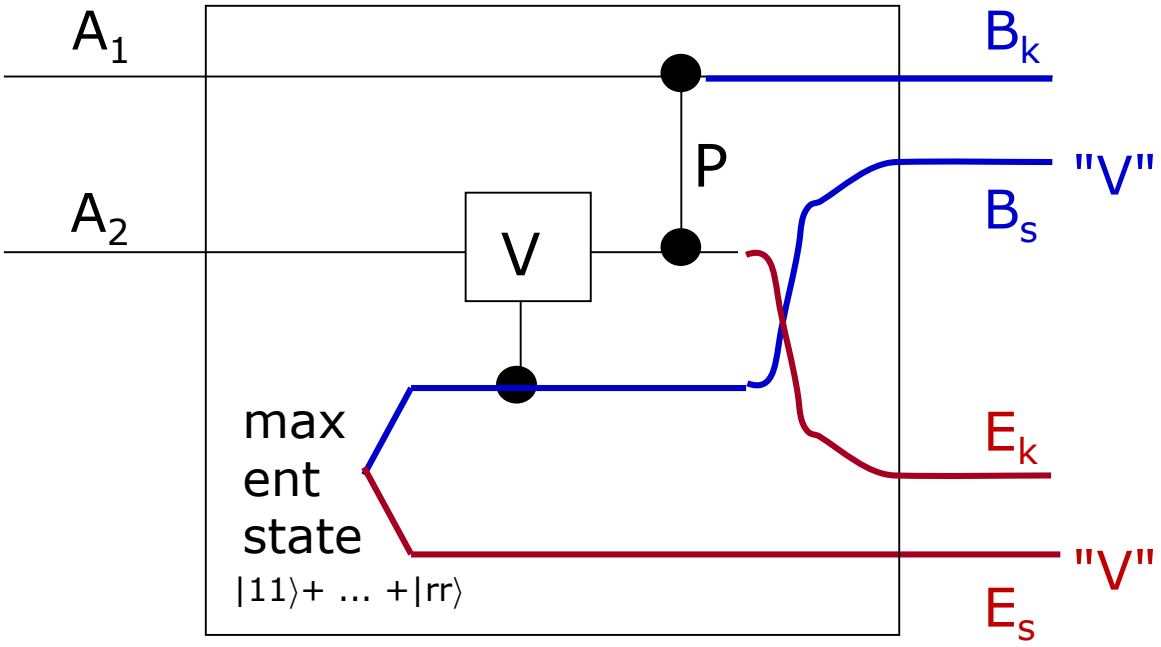
  In particular:

  - there are two channels each with no asymptotic zero-error classical capacity, but has positive joint zero-error quantum capacity (superactivation of all 3 capacities)

  - for every n, there is a channel with no n-shot zero-error quantum capacity, but positive asymptotic zero-error quantum capacity.

The non-commutative graph of N_d:

Duan-Severini-Winter & Weaver



Kraus operator for each k and V:

$$(I \otimes \langle k|)P(I \otimes V)\otimes V\rangle = (Z_k \otimes \langle k|V)\otimes|V\rangle$$

$G(N_d)$ = span{$E^\dagger F$: E,F Kraus ops}

= span{$Z_{k-l}\otimes V^\dagger|l\rangle\langle k|V : 1 \leq l,k \leq d, V \in$ 2-design}

The non-commutative graph of N_d:

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$G(N_d) = \text{span}\{E^\dagger F: E,F \text{ Kraus ops}\}$

$= \text{span}\{Z_{k-l} \otimes V^\dagger |l\rangle\langle k| V : 1 \leq l,k \leq d,\ V \in \text{2-design}\}$

Shirokov-Shulman 2015:

If G(N) constains all matrices diagonal in some basis, or if G(N) is closed under multiplication, then N cannot be superactivated.

NB $Z \otimes I \notin G(N_d)$, $Z \otimes Z$, $I \otimes Z \in G(N_d)$, so, neither condition holds, so cannot conclude $Q_0(N_d) = 0$.

Open problem: can other channels activate N_d ?

cf (regular) $Q(N_d \otimes \text{50/50 erasure channel}) \approx O(\log d)$.

Lemma (Cubitt-Smith-2012):

One can transmit quantum data through 1 use of a q channel N

iff $\exists\, |\alpha\rangle, |\beta\rangle$ s.t.

$$\mathrm{tr}\,[N(|\alpha\rangle\langle\alpha|)\, N(|\beta\rangle\langle\beta|)] = 0 \qquad (1)$$

$$\mathrm{tr}\,[N(|\alpha+\beta\rangle\langle\alpha+\beta|)\, N(|\alpha-\beta\rangle\langle\alpha-\beta|)] = 0 \qquad (2)$$

where $|\alpha \pm \beta\rangle = (|\alpha\rangle \pm |\beta\rangle)/\sqrt{2}$

Proof sketch for $Q_0(N_d) = 0$:

- characterize $|\alpha\rangle$, $|\beta\rangle$ satisfying condition (1) for 1 use of N_d
- generalize to $N_d^{\otimes n}$
- show condition (2) must fail

$\therefore\ \forall n$, no quantum data can be transmitted through $N_d^{\otimes n}$

Lemma 1:

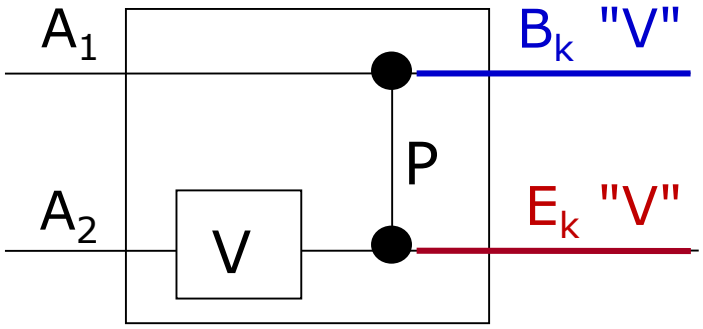
Let $|\alpha\rangle=\sum_{i=1}^{d} |i\rangle\ |\alpha_i\rangle$, $|\beta\rangle =\sum_{i=1}^{d} |i\rangle\ |\beta_i\rangle$.

Then, $\mathrm{tr}\,[N_d(|\alpha\rangle\langle\alpha|)\ N_d(|\beta\rangle\langle\beta|)] = 0$ iff $\forall_i\ |\alpha_i\rangle|\beta_i\rangle = 0$.

i.e., no overlap in the computation basis in system A_1.

Pf: for each V, let $|\alpha'\rangle = P(I \otimes V)|\alpha\rangle$; $\rho^V = \mathrm{tr}_2\ |\alpha'\rangle\langle\alpha'|$
$|\beta'\rangle = P(I \otimes V)|\beta\rangle$; $\sigma^V = \mathrm{tr}_2\ |\beta'\rangle\langle\beta'|$.

$\mathrm{tr}\,[N_d(|\alpha\rangle\langle\alpha|)\ N_d(|\beta\rangle\langle\beta|)] = 0$
iff $\forall_V\ \mathrm{tr}\ \rho^V \sigma^V = 0$
iff $E_V\ \mathrm{tr}\ \rho^V \sigma^V = 0$
$\|$
$\langle x|A|x\rangle$



where $|x\rangle = \sum_{i=1}^{d} |i\rangle|\alpha_i\rangle|\beta_i^c\rangle$,
$A = M\otimes(I - \Phi) + |\nu\rangle\langle\nu|\otimes\Phi$, $M>0$, $|\nu\rangle = 1/\sqrt{d}\ \sum_{i=1}^{d} |i\rangle$

If $|x\rangle \in \mathrm{null}(A)$, $\mathrm{tr}_2|x\rangle\langle x| \propto \Phi$, a contradiction unless $|x\rangle = 0$.

Lemma n:

Let $|\alpha\rangle=\sum_{i1,..,in} |i_1...i_n\rangle|\alpha_{i1,..,in}\rangle$, $|\beta\rangle =\sum_{i1,..,in} |i_1...i_n\rangle|\beta_{i1,..,in}\rangle$.

Then $\mathrm{tr}\,[N_d^{\otimes n} (|\alpha\rangle\langle\alpha|) N_d^{\otimes n} (|\beta\rangle\langle\beta|)] = 0$

iff $\forall_{i1,..,in} |\alpha_{i1,..,in}\rangle|\beta_{i1,..in}\rangle = 0.$

Pf: similar to 1-shot case:

$\mathrm{tr}\,[N_d^{\otimes n} (|\alpha\rangle\langle\alpha|) N_d^{\otimes n} (|\beta\rangle\langle\beta|)] = 0$

iff $\langle x|A^{\otimes n}|x\rangle = 0$

where $|x\rangle = \sum_{i1,...,in} |i_1...i_n\rangle\ |\alpha_{i1,..,in}\rangle|\beta_{i1,...,in}{}^c\rangle$,

$A = M\otimes(I - \Phi) + |\nu\rangle\langle\nu|\otimes\Phi$, $M>0$, $|\nu\rangle = 1/\sqrt{d}\ \sum_{i=1}^{d} |i\rangle$

So, $0 = \langle x|A^{\otimes n}|x\rangle \geq$ constant $* \langle x|I^{\otimes n}\otimes(I - \Phi)^{\otimes n}|x\rangle$

So, $0 = \langle x|I^{\otimes n}\otimes(I - \Phi)^{\otimes n}|x\rangle$

Let $m = \mathrm{tr}_1|x\rangle\langle x|$. So, $m \geq 0$, PPT, and $\mathrm{tr}\, m(I - \Phi)^{\otimes n} = 0$.

By Yu-Duan-Ying-2014, $m = 0$. So, $\forall_{i1,..,in} |\alpha_{i1,..,in}\rangle|\beta_{i1,..in}\rangle = 0.$

Lemma n:

Let $|\alpha\rangle = \sum_{i1,..,in} |i_1 \ldots i_n\rangle |\alpha_{i1,..,in}\rangle$, $|\beta\rangle = \sum_{i1,..,in} |i_1 \ldots i_n\rangle |\beta_{i1,..,in}\rangle$.

Then $\mathrm{tr}\,[N_d^{\otimes n}(|\alpha\rangle\langle\alpha|)\, N_d^{\otimes n}(|\beta\rangle\langle\beta|)] = 0$

iff $\forall_{i1,..,in}\ |\alpha_{i1,..,in}\rangle |\beta_{i1,..in}\rangle = 0.$

If $\mathrm{tr}\,[N_d^{\otimes n}(|\alpha\rangle\langle\alpha|)\, N_d^{\otimes n}(|\beta\rangle\langle\beta|)] = 0$ (1)

& $\mathrm{tr}\,[N_d^{\otimes n}(|\alpha+\beta\rangle\langle\alpha+\beta|)\, N_d^{\otimes n}(|\alpha-\beta\rangle\langle\alpha-\beta|)] = 0$ (2)

For each $i_1, \ldots, i_n$,

at most one of $|\alpha_{i1,..,in}\rangle$, $|\beta_{i1,..in}\rangle$ nonzero, and

at most one of $|\alpha_{i1,..,in}\rangle + |\beta_{i1,..in}\rangle$, $|\alpha_{i1,..,in}\rangle - |\beta_{i1,..in}\rangle$ nonzero.

$\therefore |\alpha_{i1,..,in}\rangle = |\beta_{i1,..in}\rangle = 0$, a contradiction.

Theorem:

For any natural number n, N_d^n cannot send any quantum data with zero error.

Further open problem

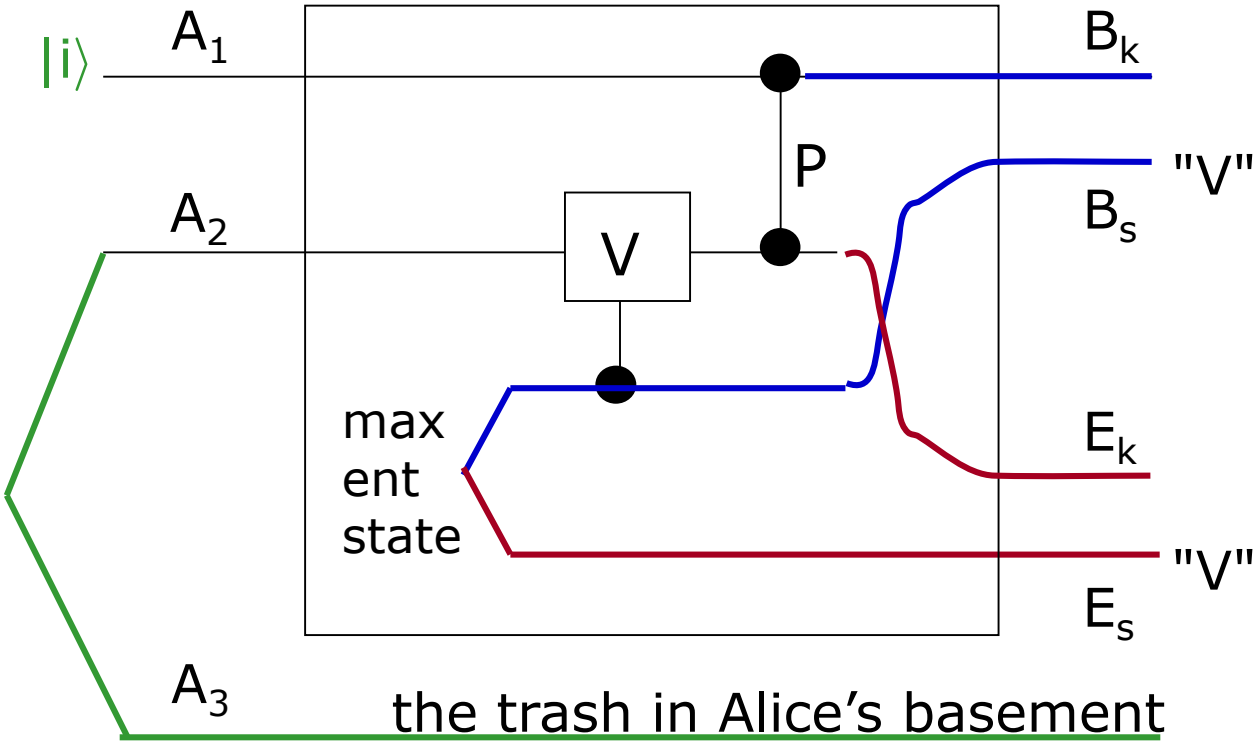
How to characterize when a quantum channel has no zero-error quantum capacity?

- Thank you!

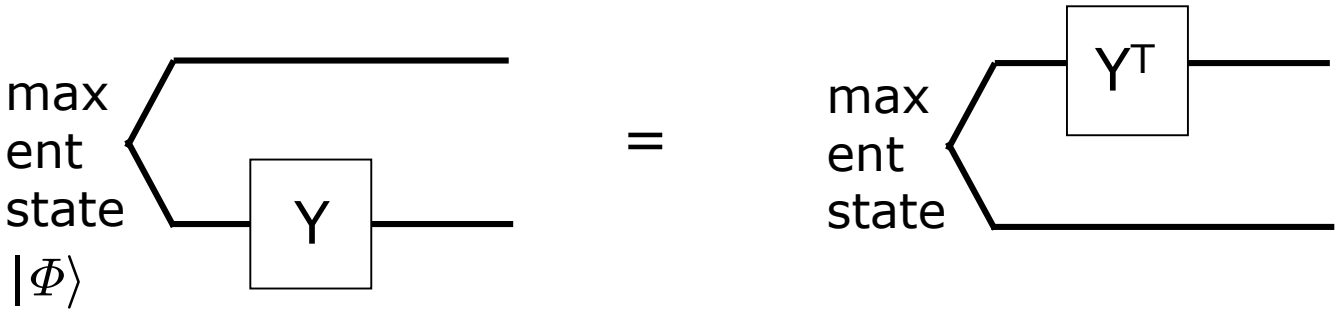
Alternative Pf from 1st principle.

Why $P_0(N_d) = \log d$:

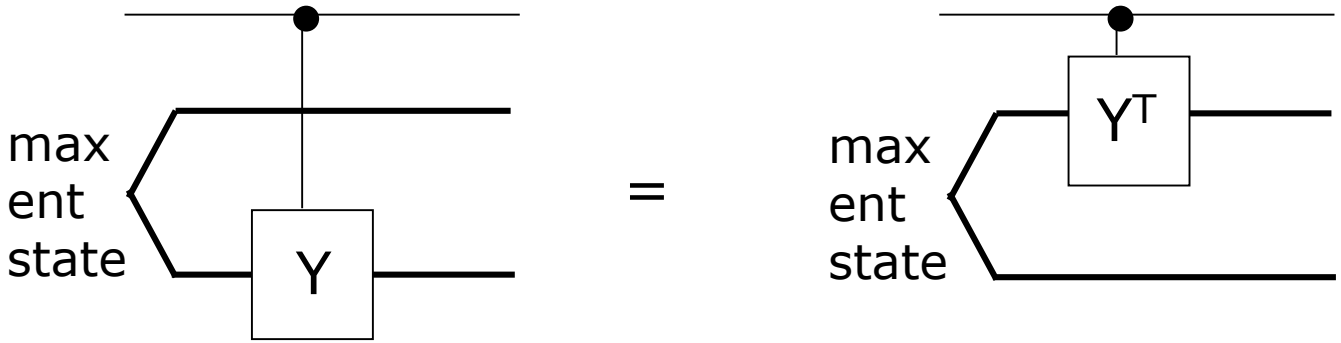
Let Alice's private message be $i \in \{1,2,\cdots,d\}$.
She inputs $|i\rangle$ into A_1, and half of a max ent state into A_2, holding the other half in A_3.



The transpose trick



$$(IY)|\Phi\rangle = (IY^T)|\Phi\rangle$$



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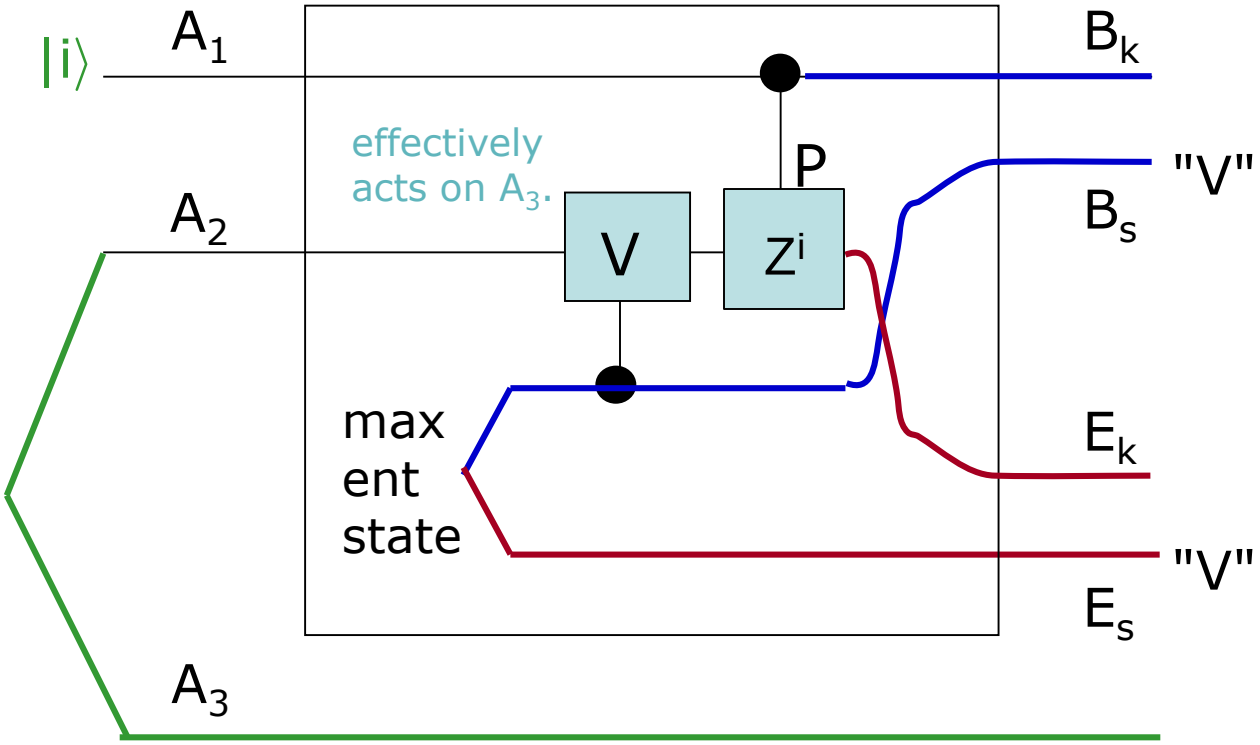

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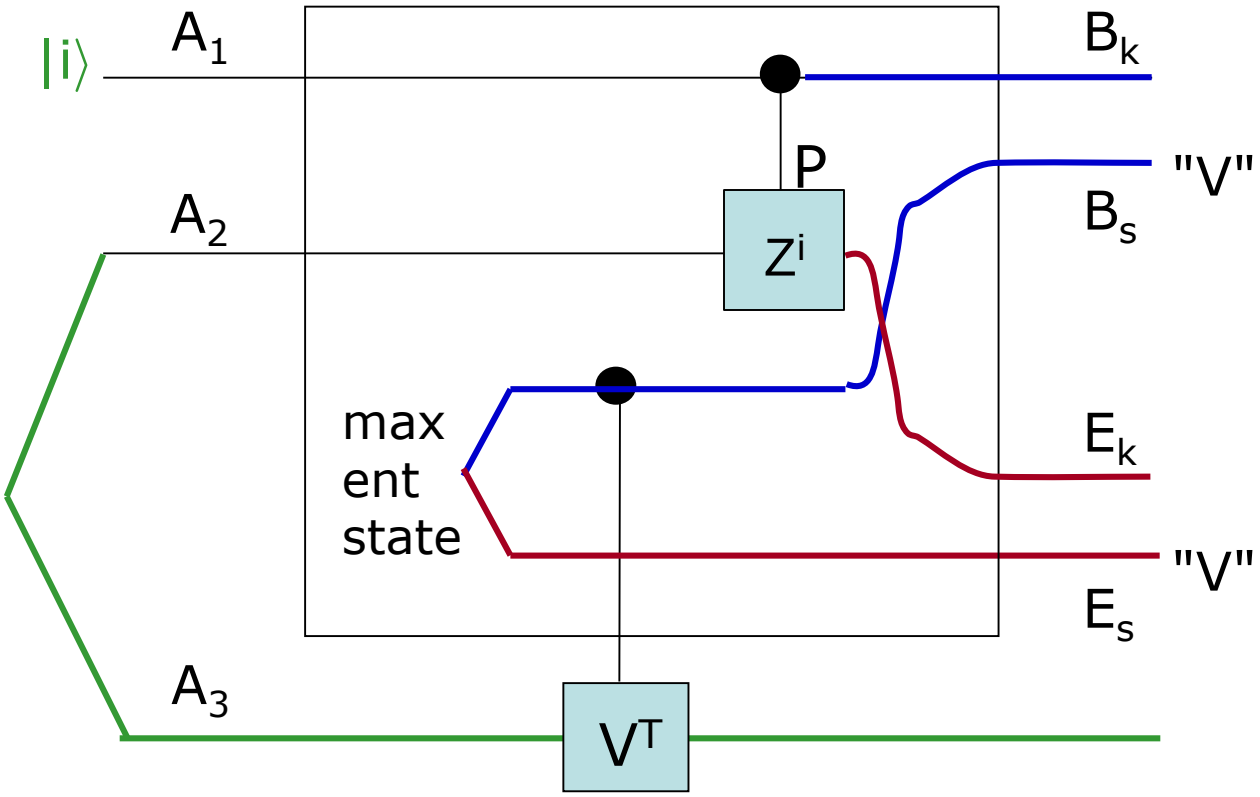

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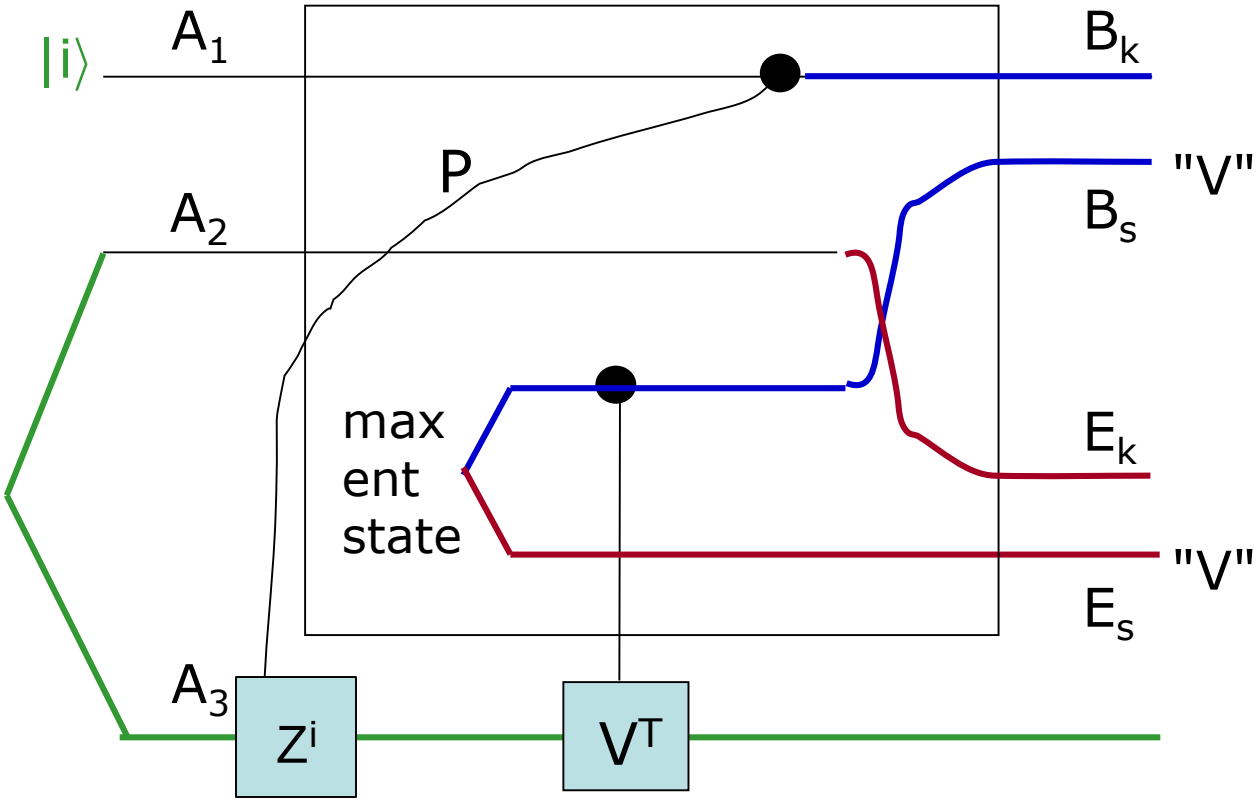

Why $P_0(N_d) = \log d$:

This an equivalent circuit in which E_k, E_s exist before i; they must be independent of i. So, $P_0(N) \geq \log d$.

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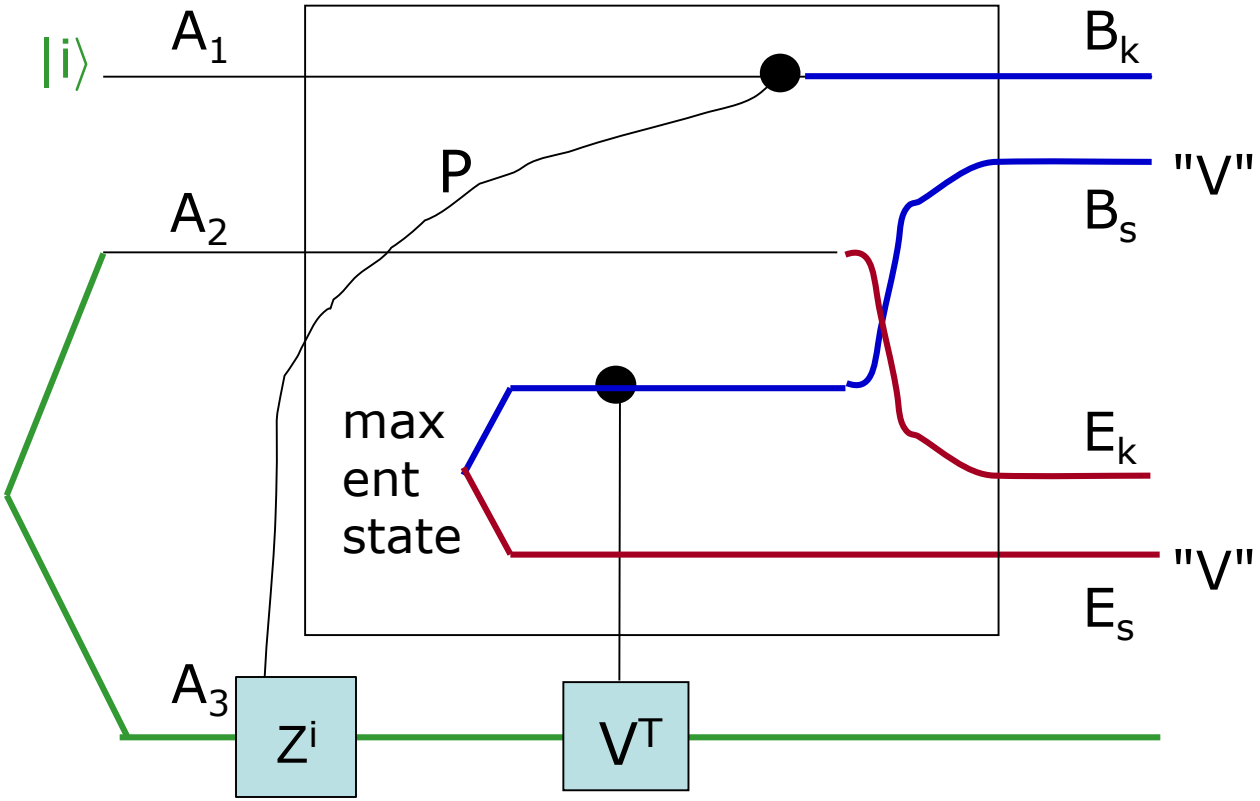
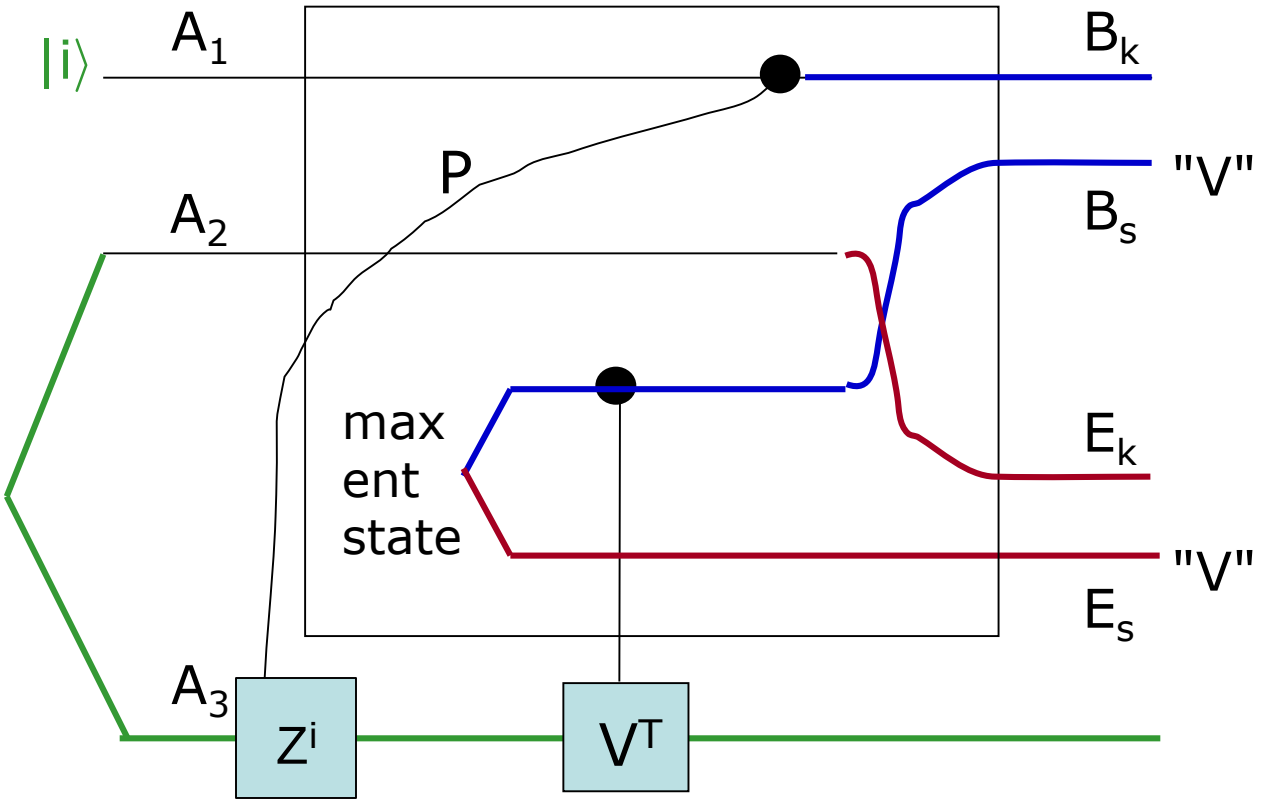
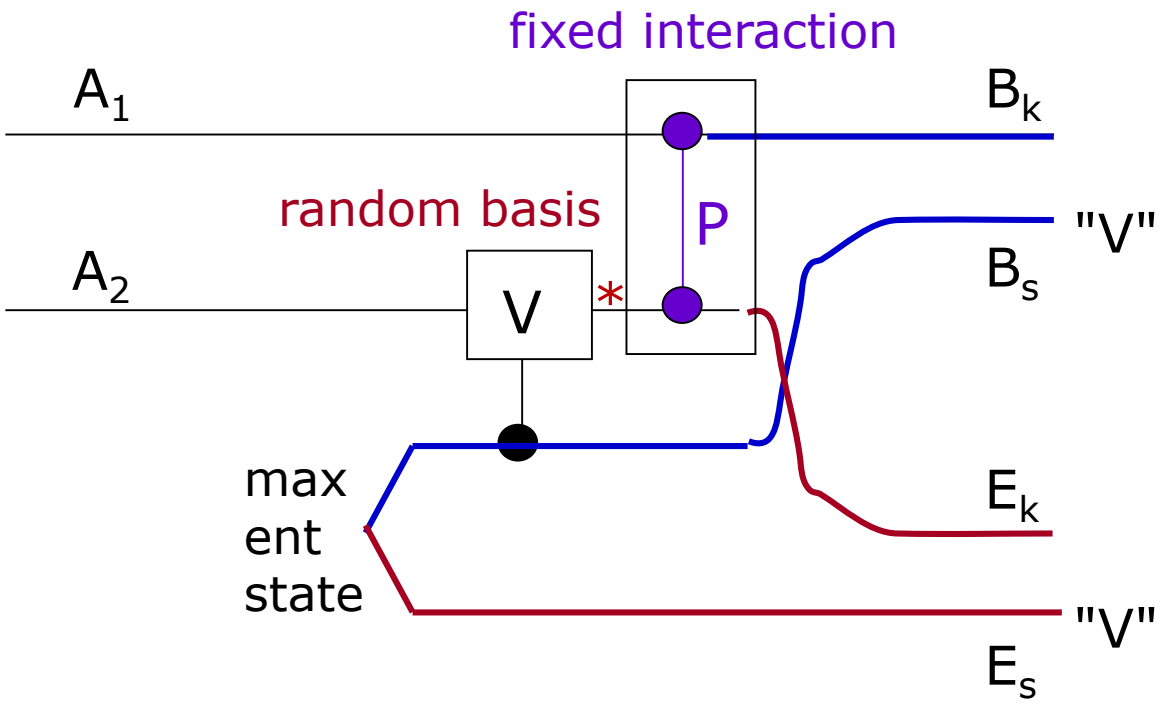


Turning privacy to coherence?

Note: B_k entangled with A_3 which is entangled with E_k – this method cannot be tweaked to create entanglement between Alice and Bob.



Intuition for small quantum capacity



B_k, E_k almost certainly very very entangled.