

Introduction to measurement-based quantum computation

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Supported by



Goals of this tutorial*

- ❑ Give some details to understand basic ingredients of measurement-based quantum computation (MBQC)
- ❑ Give pointers to related development/ application (fewer details)
- ❑ Will point out related talks in this conference
- ❑ Give some open problems

*This tutorial assumes little prior knowledge

Review papers

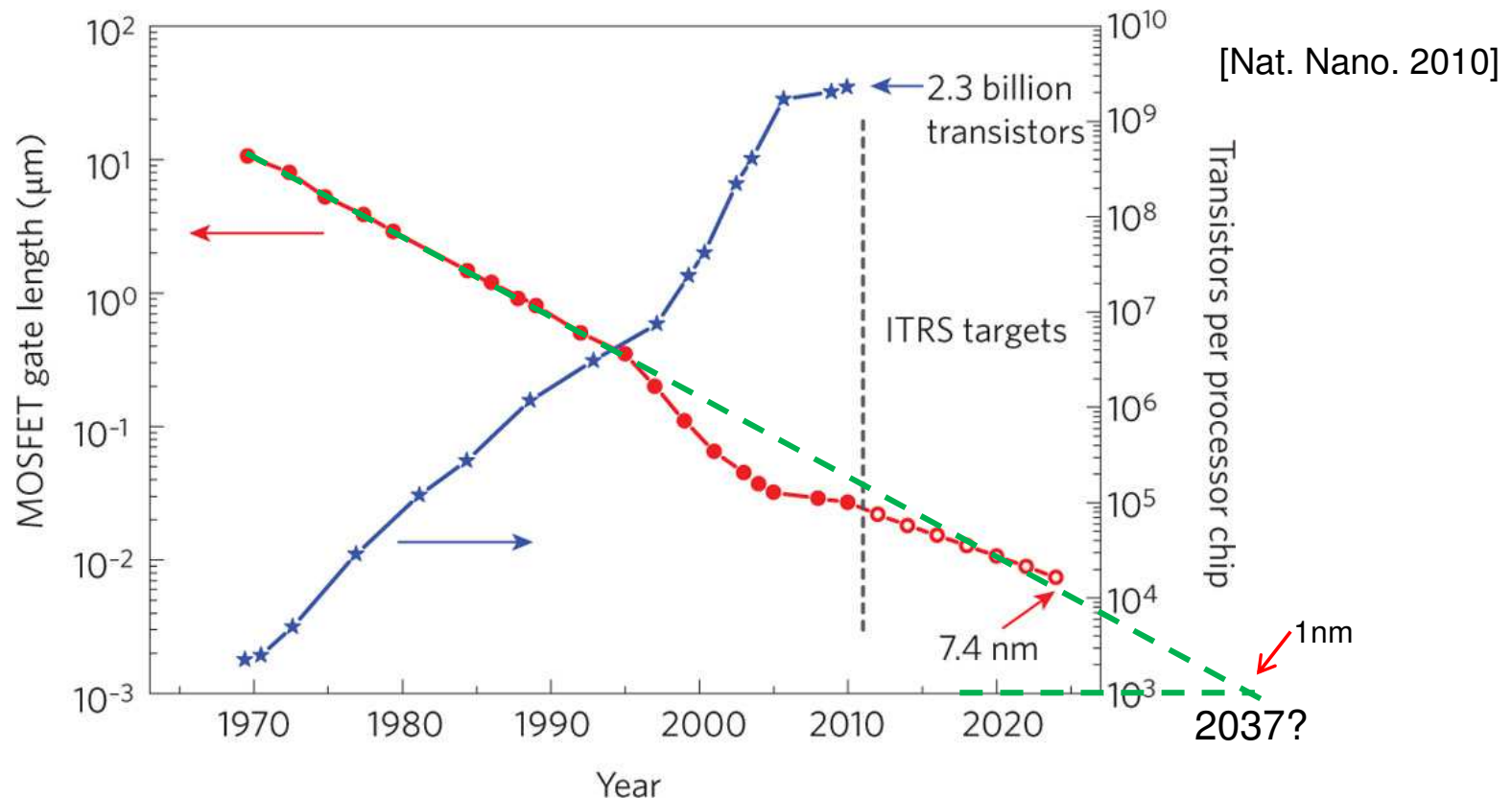
H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf & M. Van den Nest, “Measurement-based quantum computation” Nat. Phys. **5**, 19 (2009)

R. Raussendorf & T-C Wei, “Quantum computation by local measurement”, [Annual Review of Condensed Matter Physics](#), 3, 239 (2012)

L.C. Kwek, Z.H. Wei & Bei Zeng. “Measurement-Based Quantum Computing with Valence-Bond-Solids”, Int. J. Mod. Phys. B 26, 123002 (2012)

Moore's Law:

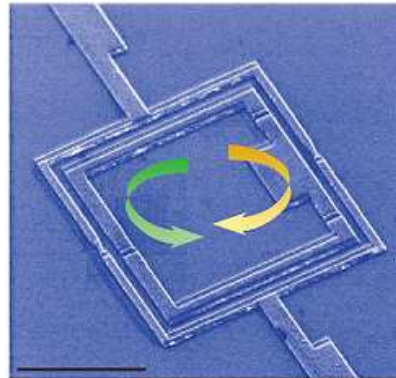
The number of transistors on a chip doubles ~every 2 years



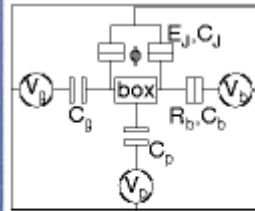
→ A transistor hits the size of a few atoms in about 20 years

→ Quantum regime is inevitable

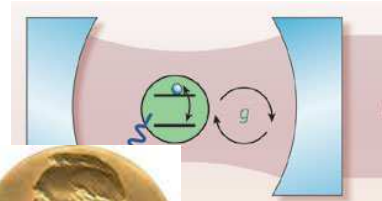
Candidate systems* for quantum computers



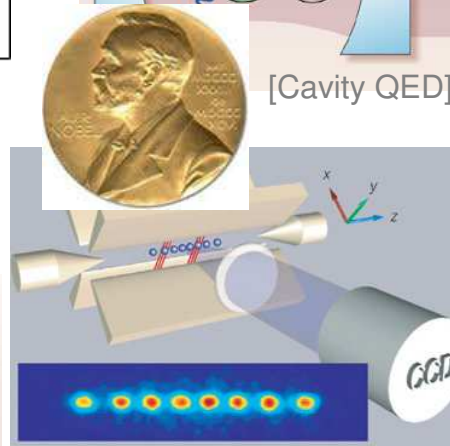
[Superconductors (Josephson junctions)]



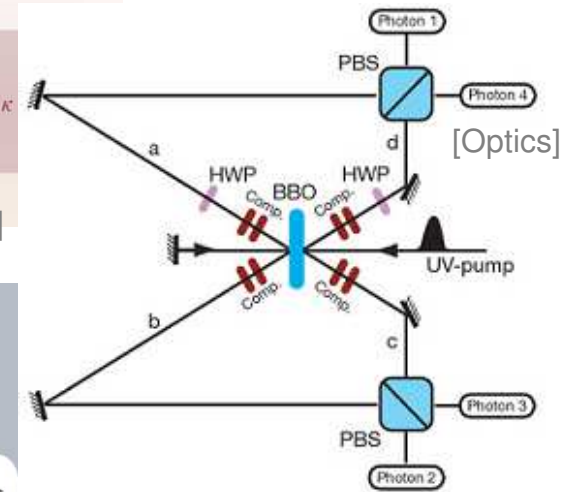
□: tunnel junction
||: capacitor



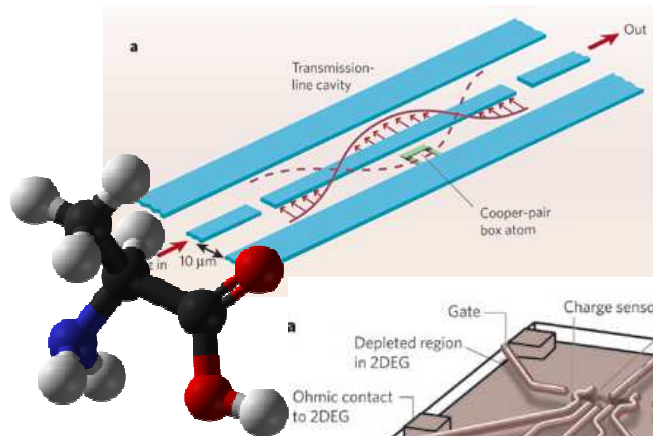
[Cavity QED]



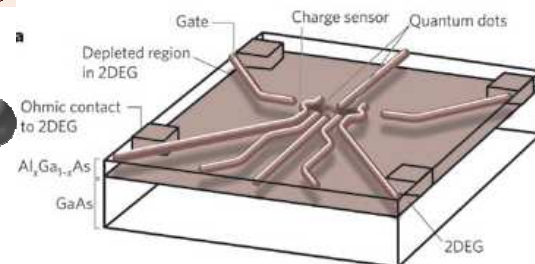
[Trapped Ions and atoms]



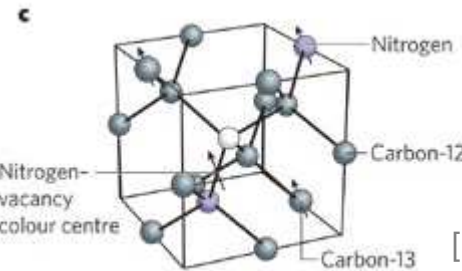
[Optics]



[NMR]



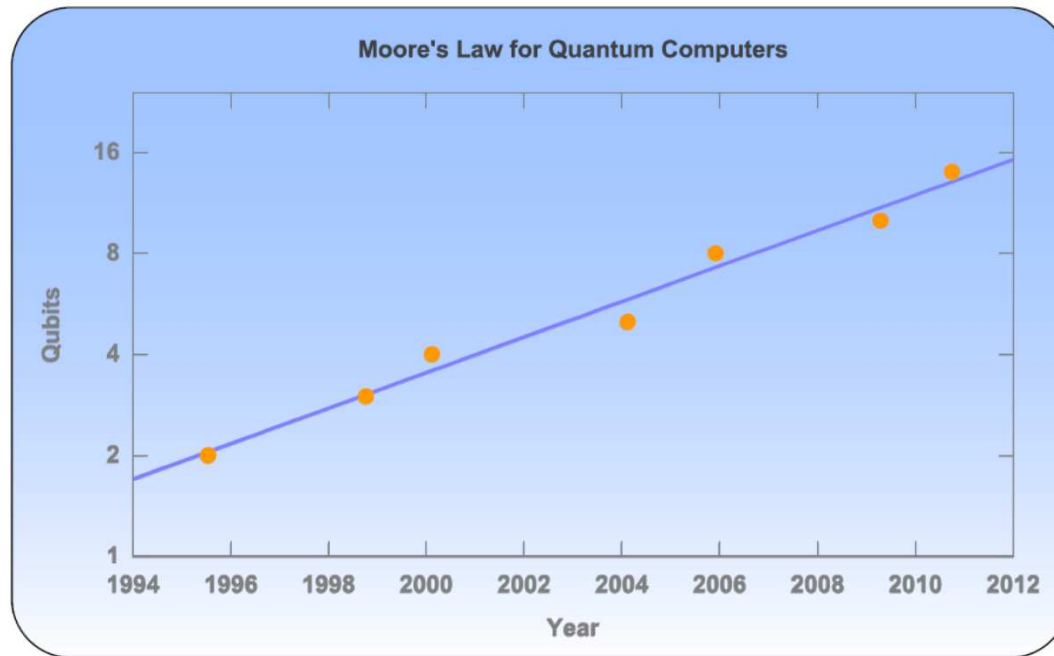
[Quantum dot]



[NV center in diamond]

*You may see many of these throughout this conference

New quantum Moore's Law?



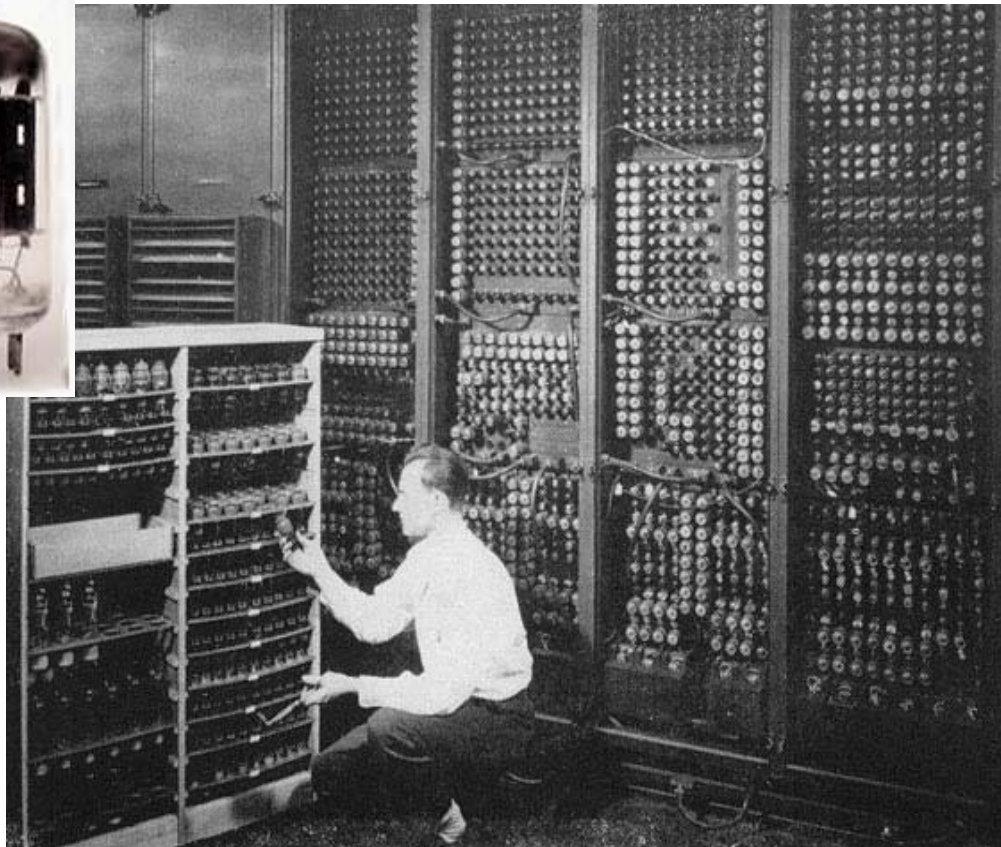
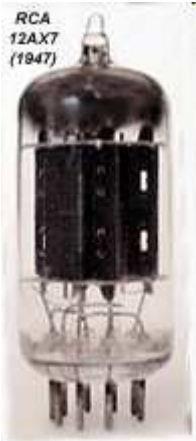
[H. Weimer]

- Number of qubits in ion trap

→ Roughly doubles every 6 years!
(may depend on physical systems)

e.g. see Nathan Langford's tutorial on circuit QED

ENIAC – first generation computer



Contained:
17,468 vacuum tubes,
7,200 crystal diodes,
1,500 relays,
70,000 resistors,
10,000 capacitors
5 million hand-soldered joints

Weighed 27 tons
About 8.5 by 3 by 100 feet
Took up 1800 square feet

20 ten-digit signed accumulators

[1946]

When will the first-generation quantum computer appear?

Quantum computation in a nutshell

- Consider a function f and a corresponding unitary U :

$$U_f : |k\rangle \otimes |0\rangle \longrightarrow |k\rangle \otimes |f(k)\rangle$$

- Exploit quantum parallelism:

$$\left(\sum_{k=0}^{2^n-1} |k\rangle \right) \otimes |0\rangle \xrightarrow{U_f} \sum_{k=0}^{2^n-1} |k\rangle \otimes |f(k)\rangle$$

- Naive measurement only gives one $f(k)$ at a time
- Good design of measurement may reveal properties of f
➔ e.g. Shor's factoring algorithm

- Factoring is hard:

180708208868740480595165616440590556627810251676940134917012702
1450056662540244048387341127590812303371781887966563182013214880

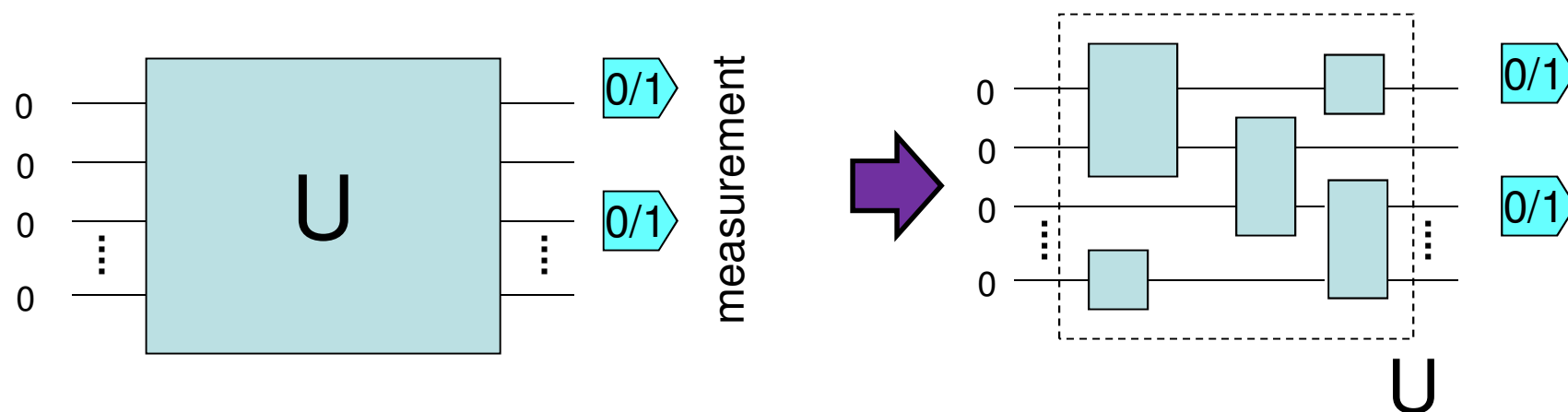
557 = (????....?) x (????....?)

=(39685999459597454290161126162883786067576449112810064832555157243)

x

(45534498646735972188403686897274408864356301263205069600999044599)

Quantum computation: Circuit model



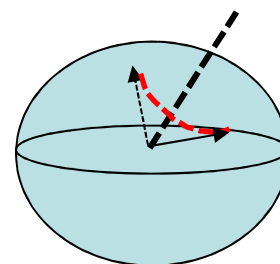
□ Building blocks

Universal gates

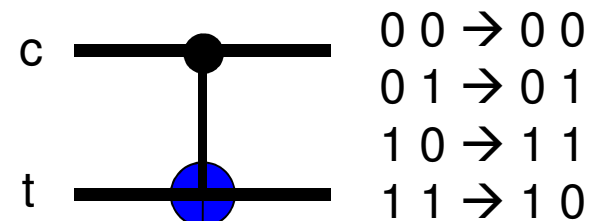
(1) One qubit gates: any rotation

(2) Two qubit gate: entangling

e.g., C-Z gate or
Controlled-NOT gate



CNOT:

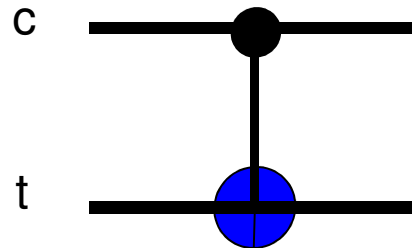


CNOT & CZ gates

CNOT:

$$\text{CNOT} = |0\rangle_c \langle 0| \otimes I_t + |1\rangle_c \langle 1| \otimes X_t$$

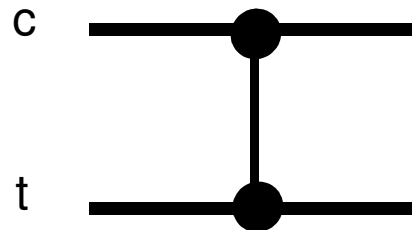
$00 \rightarrow 00$
 $01 \rightarrow 01$
 $10 \rightarrow 11$
 $11 \rightarrow 10$



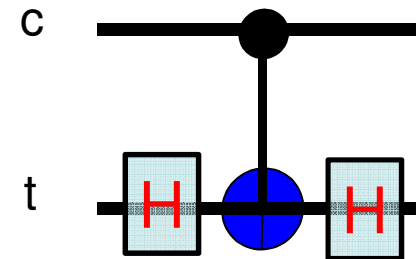
CZ:

$$\text{CZ} = |0\rangle_c \langle 0| \otimes I_t + |1\rangle_c \langle 1| \otimes Z_t$$

$00 \rightarrow 00$
 $01 \rightarrow 01$
 $10 \rightarrow 10$
 $11 \rightarrow -11$

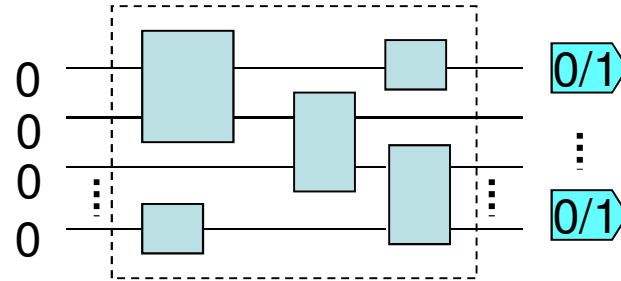


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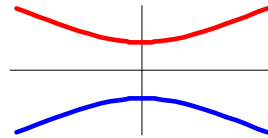


(Models of) Quantum Computation

□ Circuit:

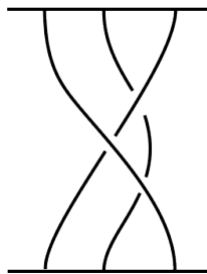


□ Adiabatic:



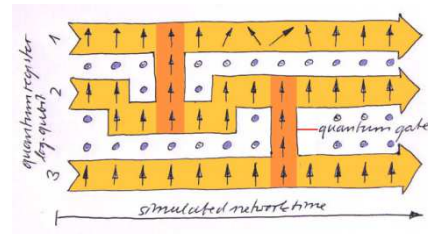
$$H(t) = \left(1 - \frac{t}{T}\right) H_{\text{initial}} + \frac{t}{T} H_{\text{final}}$$

□ Topological:



using braiding of anyons to
simulate quantum gates

□ Measurement-based:



local measurement is
the only operation needed

Outline

I. Introduction

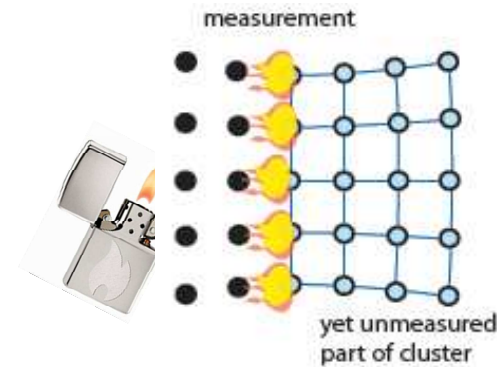
II. One-way cluster-state (or measurement-based) quantum computers

III. Other entangled resource states: Affleck-Kennedy-Lieb-Tasaki (AKLT) family

IV. Summary

Now focus on measurement-based
(or one-way) quantum computer:

which can “simulate” unitary evolution



Unitary operation by measurement?

□ Intuition: entanglement as resource!

❖ Controlled-Z (CZ) gate from Ising interaction

$$CZ_{12} = e^{-i\frac{\pi}{4}(1-\sigma_Z^{(1)})(1-\sigma_Z^{(2)})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



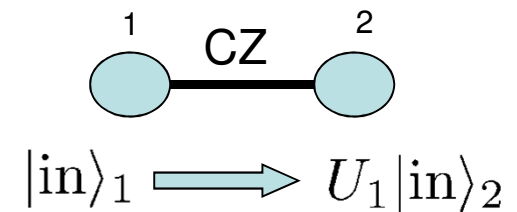
❖ Entanglement is generated:

$$(a|0\rangle + b|1\rangle) |+\rangle \xrightarrow{CZ} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$

Unitary operation by measurement?

□ Intuition: entanglement as resource!

$$(a|0\rangle + b|1\rangle)|+\rangle \xrightarrow{\text{CZ}} |\psi\rangle = a|0\rangle|+\rangle + b|1\rangle|-\rangle$$



measurement

❖ Measurement on 1st qubit in basis

$$|\pm\xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$$

with outcome denoted by $\pm = (-1)^s$

$$\begin{array}{cc} \cos(\xi)\sigma_x + \sin(\xi)\sigma_y & \\ \text{III} & \text{III} \\ X & Y \end{array}$$

➔ Second qubit becomes

$${}_1\langle\pm\xi|\psi\rangle_{12} \sim a e^{i\xi/2}|+\rangle_2 \pm b e^{-i\xi/2}|-\rangle_2 = H e^{i\xi Z/2} Z^s (a|0\rangle_2 + b|1\rangle_2)$$

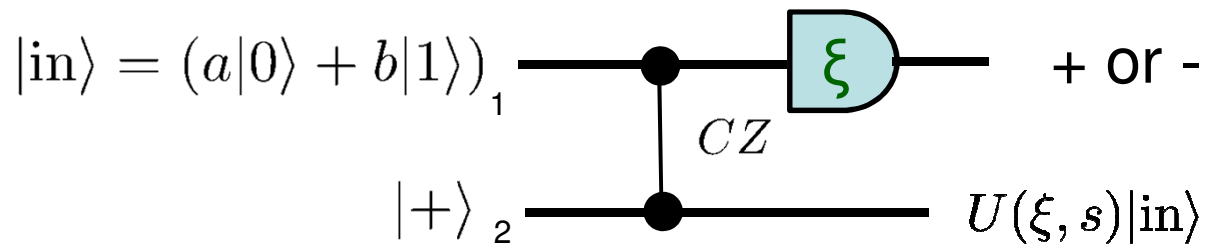
➔ A unitary gate is induced: $U(\xi, s) \equiv H e^{i\xi Z/2} Z^s$

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Simulating arbitrary one-qubit gates

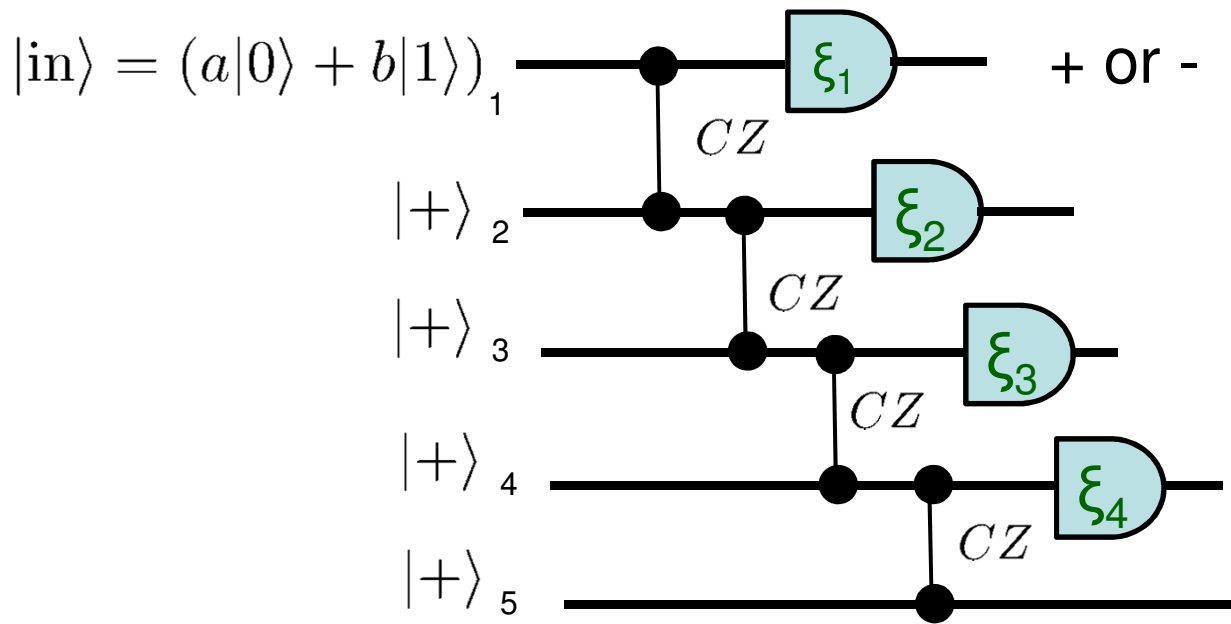
□ In terms of circuit:

[Raussendorf & **Wei**, Ann Rev Cond-Mat '12]



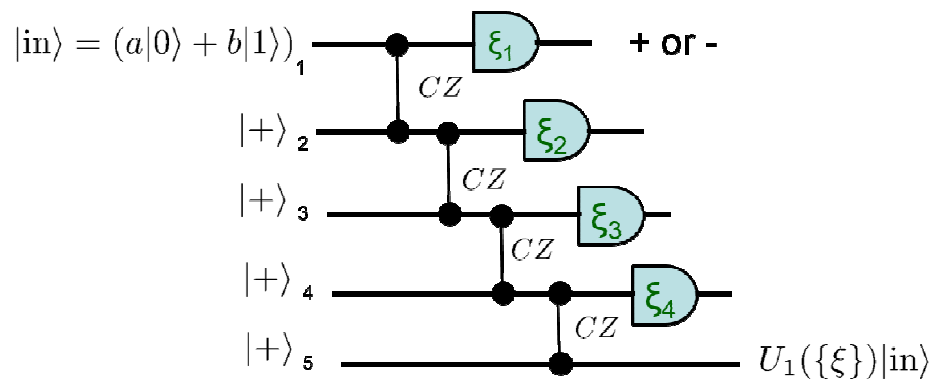
$$U(\xi, s) = H e^{i\xi Z/2} Z^s$$

□ Can cascade this a few times:



$$U_1(\{\xi\}) = \prod_{i=1}^4 U(\xi_i, s_i)$$

Example: arbitrary one-qubit gate



$$U_1(\{\xi\}) = \prod_{i=1}^4 U(\xi_i, s_i) \quad U(\xi, s) = H e^{i\xi Z/2} Z^s$$

□ Consider: $\xi_i=0$ & construct arbitrary rotation

$$U_1(\{\xi\}) = (H e^{i\xi_4 Z/2} Z^{s_4}) (H e^{i\xi_3 Z/2} Z^{s_3}) (H e^{i\xi_2 Z/2} Z^{s_2}) (H Z^{s_1})$$

□ Propagating Z's to left and use $HZH=X$:

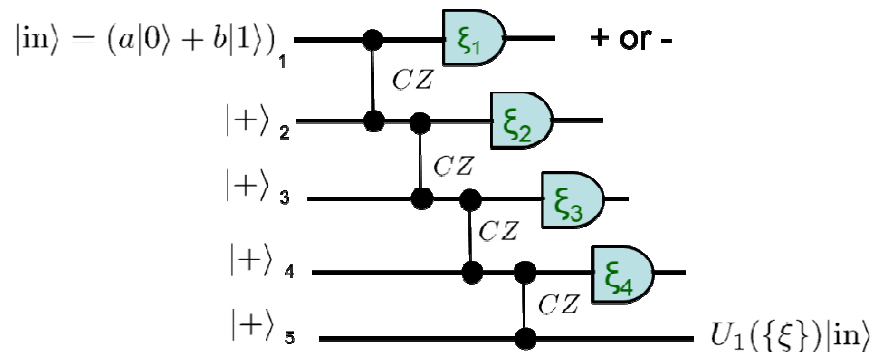
$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{i(-1)^{s_1+s_3}\xi_4 X/2} e^{i(-1)^{s_2}\xi_3 Z/2} e^{i(-1)^{s_1}\xi_2 X/2}$$

□ Take $\xi_2 = -(-1)^{s_1}\gamma$, $\xi_3 = -(-1)^{s_2}\beta$, $\xi_4 = -(-1)^{s_1+s_3}\alpha$

we realize an Euler rotation, up to byproduct Z, X operators:

$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{-i\alpha X/2} e^{-i\beta Z/2} e^{-i\gamma X/2}$$

Comments



$$U_1(\{\xi\}) = \prod_{i=1}^4 U(\xi_i, s_i) \quad U(\xi, s) = H e^{i\xi Z/2} Z^s$$

□ Consider: $\xi_i=0$, & construct arbitrary rotation

□ Take $\xi_2 = -(-1)^{s_1}\gamma$, $\xi_3 = -(-1)^{s_2}\beta$, $\xi_4 = -(-1)^{s_1+s_3}\alpha$

we realize an Euler rotation, up to byproduct Z, X operators:

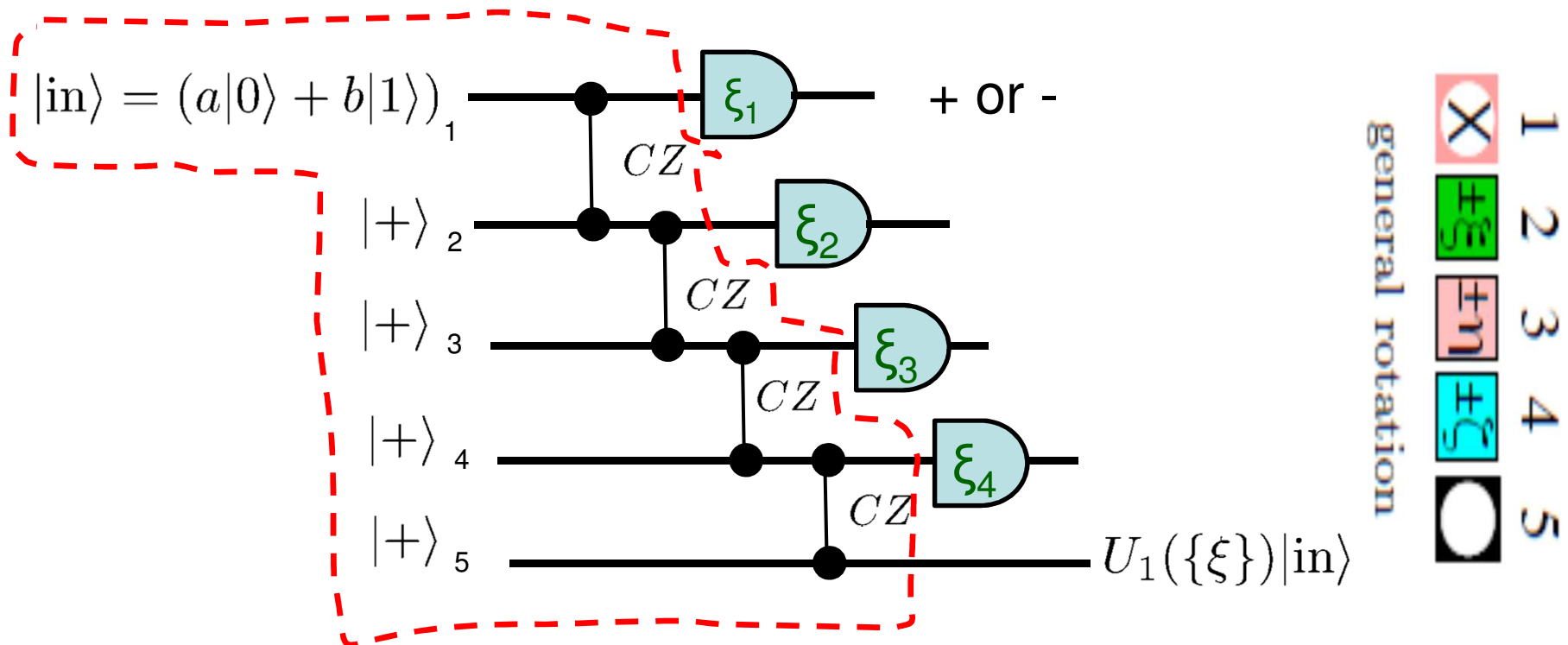
$$U_1(\{\xi\}, \{s\}) = Z^{s_1+s_3} X^{s_2+s_4} e^{-i\alpha X/2} e^{-i\beta Z/2} e^{-i\gamma X/2}$$

→ Note: measurement basis can depend on prior results

→ Byproduct operators $Z^{s_1+s_3} X^{s_2+s_4}$ can be absorbed by modifying later measurement basis

→ Byproduct operators on final measurement in Z basis (readout) can be easily taken into account (only X flips 0/1)

Linear cluster state: resource for simulating arbitrary one-qubit gates



- May as well take $|\text{in}\rangle = |+\rangle$ the whole state before measurement ξ 's is a highly entangled state \rightarrow 1D cluster state

Simulating CNOT by measurement

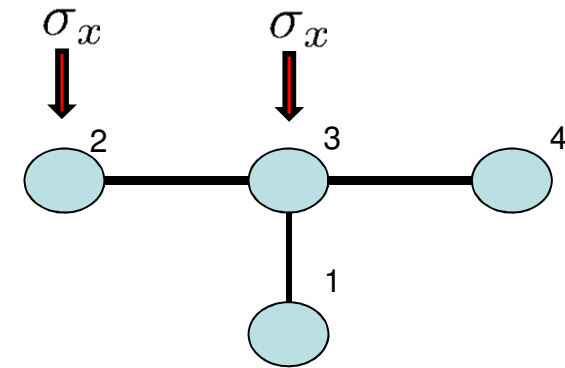
- Consider initial state

$$(a|0\rangle + b|1\rangle)_1 (c|0\rangle + d|1\rangle)_2 |+\rangle_3 |+\rangle_4$$

$$\xrightarrow{CZ_{23} CZ_{13} CZ_{34}} |\psi\rangle_{1234}$$

$$|\psi\rangle_{1234} = |0\rangle_3 (a|0\rangle_1 + b|1\rangle_1) (c|0\rangle_2 + d|1\rangle_2) |+\rangle_4$$

$$+ |1\rangle_3 (a|0\rangle_1 - b|1\rangle_1) (c|0\rangle_2 - d|1\rangle_2) |-\rangle_4$$



$$|\psi_{\text{in}}\rangle_{12} \xrightarrow{\text{red arrow}} \text{CNOT} |\psi_{\text{in}}\rangle_{14}$$

- Measurement on 2nd and 3rd qubits in basis $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$

If outcome=++: an effective CNOT applied:

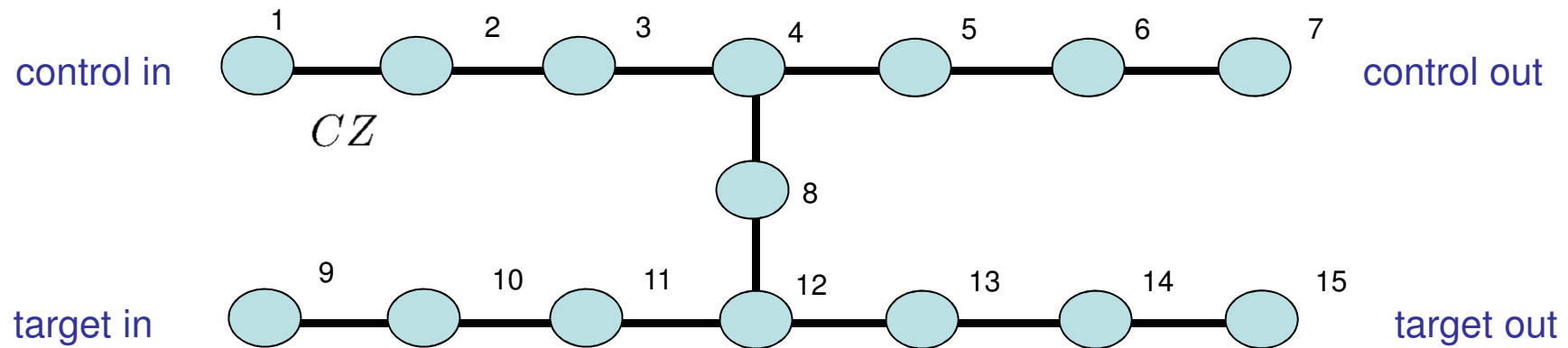
$$|\psi\rangle_{14} = {}_{23}\langle ++ | \psi \rangle_{1234} \sim \text{CNOT}_{14} (a|0\rangle_1 + b|1\rangle_1) (c|0\rangle_4 + d|1\rangle_4)$$

Can show: $|\psi_{\text{out}}\rangle \sim Z_1^{s_2} X_4^{s_3} Z_4^{s_2} \text{CNOT}_{14} |\text{in}\rangle_{14}$

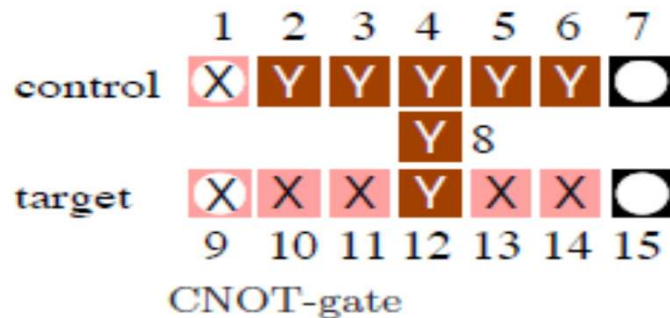
- Note the action of CZ gates can be pushed up front
(a 4-qubit “cluster” state can be used to simulating CNOT)

CNOT gate: symmetric design

[Raussendorf & Briegel PRL 01']



→ The following measurement pattern simulates CNOT gate
(via entanglement between wires)



Q: how do I know it implements CNOT?
byproduct operators=?

Ans: see Theorem I in Raussendorf, Browne & Briegel PRA '03
(generalization to qudit: Zhou et al. PRA '03)

Theorem 1. Let $\mathcal{C}(g) = \mathcal{C}_I(g) \cup \mathcal{C}_M(g) \cup \mathcal{C}_O(g)$ with $\mathcal{C}_I(g) \cap \mathcal{C}_M(g) = \mathcal{C}_I(g) \cap \mathcal{C}_O(g) = \mathcal{C}_M(g) \cap \mathcal{C}_O(g) = \emptyset$ be a cluster for the simulation of a gate g , realizing the unitary transformation U , and $|\phi\rangle_{\mathcal{C}(g)}$ the cluster state on the cluster $\mathcal{C}(g)$.

Suppose the state $|\psi\rangle_{\mathcal{C}(g)} = P_{\{s\}}^{(\mathcal{C}_M(g))}(\mathcal{M}) |\phi\rangle_{\mathcal{C}(g)}$ obeys the $2n$ eigenvalue equations

$$\sigma_x^{(\mathcal{C}_I(g), i)} (U \sigma_x^{(i)} U^\dagger)^{(\mathcal{C}_O(g))} |\psi\rangle_{\mathcal{C}(g)} = (-1)^{\lambda_{x,i}} |\psi\rangle_{\mathcal{C}(g)}, \quad (61)$$

$$\sigma_z^{(\mathcal{C}_I(g), i)} (U \sigma_z^{(i)} U^\dagger)^{(\mathcal{C}_O(g))} |\psi\rangle_{\mathcal{C}(g)} = (-1)^{\lambda_{z,i}} |\psi\rangle_{\mathcal{C}(g)},$$

with $\lambda_{x,i}, \lambda_{z,i} \in \{0, 1\}$ and $1 \leq i \leq n$.

Then, on the cluster $\mathcal{C}(g)$ the gate g acting on an arbitrary quantum input state $|\psi_{\text{in}}\rangle$ can be realized according to Scheme 1 with the measurement directions in $\mathcal{C}_M(g)$ described by $\mathcal{M}^{(\mathcal{C}_M(g))}$ and the measurements of the qubits in $\mathcal{C}_I(g)$ being σ_x measurements. Thereby, the input and output state in the simulation of g are related via

$$|\psi_{\text{out}}\rangle = U U_\Sigma |\psi_{\text{in}}\rangle, \quad (62)$$

where U_Σ is a byproduct operator given by

$$U_\Sigma = \bigotimes_{(\mathcal{C}_I(g) \ni i) = 1}^n (\sigma_z^{[i]})^{s_i + \lambda_{x,i}} (\sigma_x^{[i]})^{\lambda_{z,i}}. \quad (63)$$

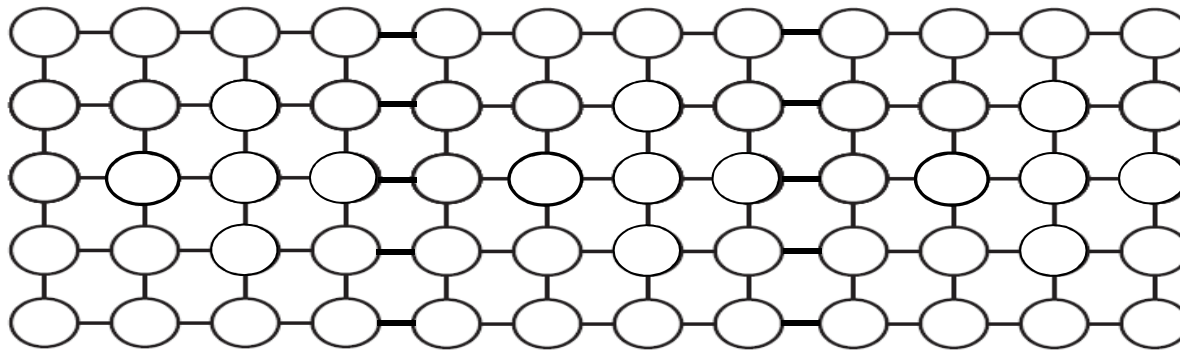
[Raussendorf, Browne
& Briegel PRA '03]

2D cluster state and graph states

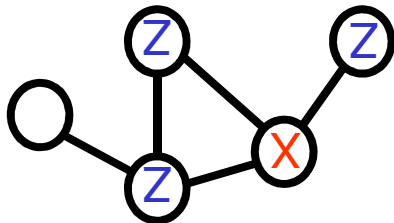
- Can be created by applying CZ gates to each pair with edge

$$|G\rangle = \bigotimes_{\langle i,j \rangle} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$$

[Raussendorf&Briegel '01]



- Cluster state: special case of general “graph” states



$$K_v |G\rangle = |G\rangle, \quad \forall \text{ vertex } v$$

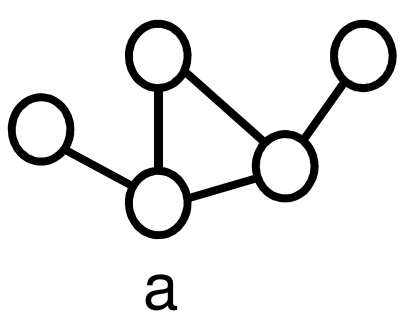
$$K_v = X_v \bigotimes_{u \in \text{Nb}(v)} Z_u$$

(can show this, using above def. of G)

→ Uniquely define the state G, also via Hamiltonian $H = - \sum_v K_v$

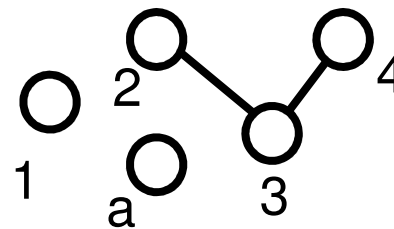
Z measurement on graph state

- The effect is just to remove the measured qubit, keeping the remaining entanglement structure



$$|\Psi_G\rangle = |0\rangle_a |\Psi_{G\setminus a}\rangle + |1\rangle_a \left(\prod_{b \in NB(a)} Z_b \right) |\Psi_{G\setminus a}\rangle$$

→ Graph after Z measurement on a:

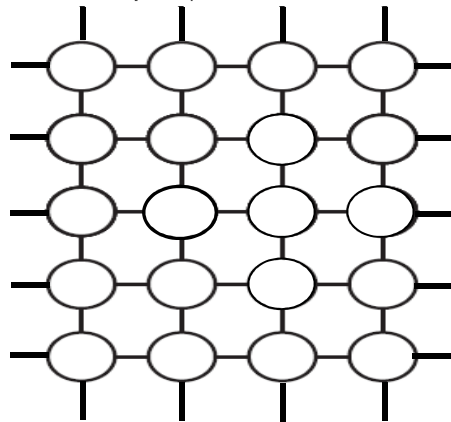


- ✓ If outcome = 0: $|0\rangle_a |+\rangle_1 |C\rangle_{234}$ $|C\rangle_{234}$: linear cluster state
- ✓ If outcome = 1: $|0\rangle_a |-\rangle_1 Z_2 Z_3 |C\rangle_{234}$

- For X & Y measurements, see [Hein, Eisert, Briegel '04, Hein et al. '06]

2D cluster state is a resource for quantum computation

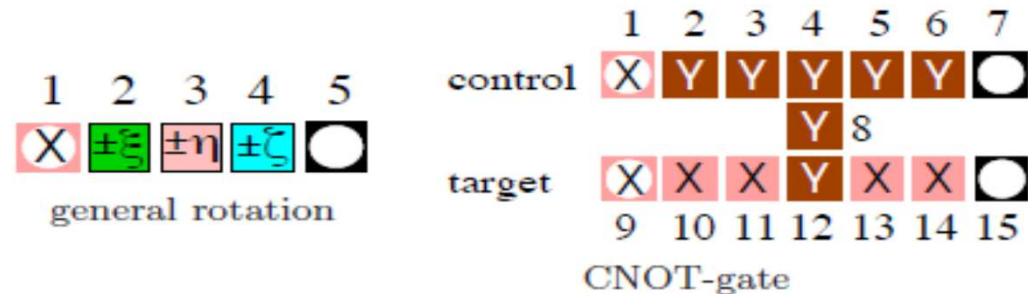
$$|C\rangle = \bigotimes_{\langle i,j \rangle} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$$



- Whole entangled state is created first (by whatever means)
- Operations needed for *universal* QC are single-qubit measurements only

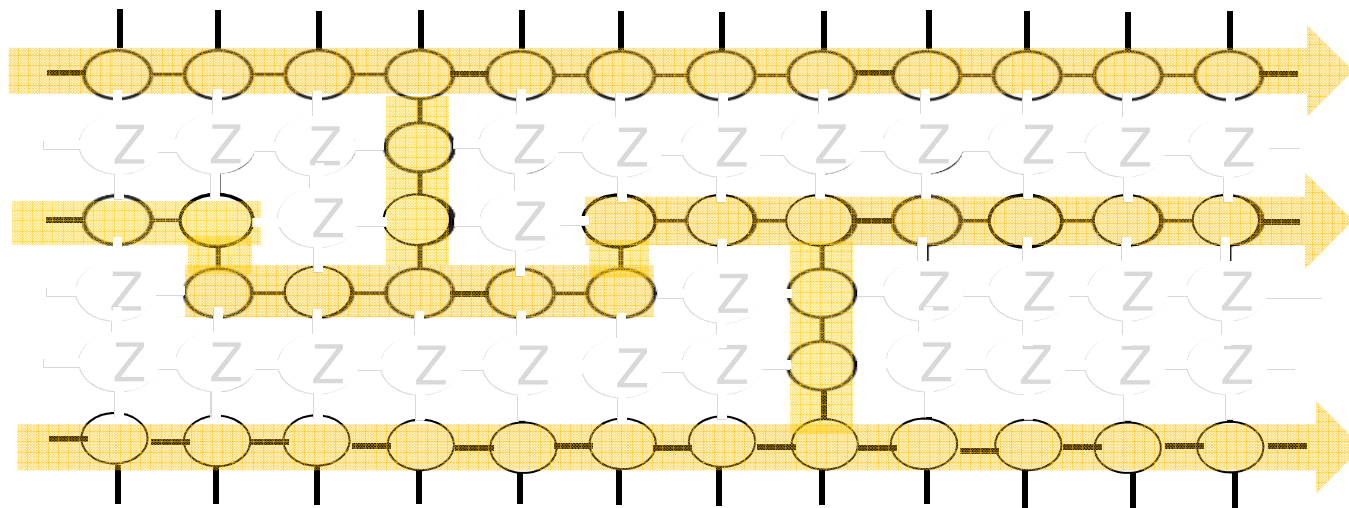
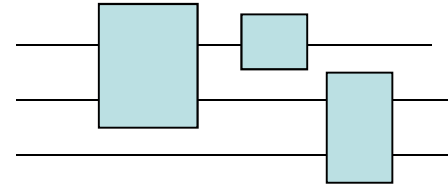
→ Pattern of measurement gives computation
(entanglement is being consumed → one-way)

→ Elementary “Lego pieces” for QC:



Cluster state for universal computation

- Carve out entanglement structure by local Z measurement



(1) Each wire simulates one-qubit evolution (gates)

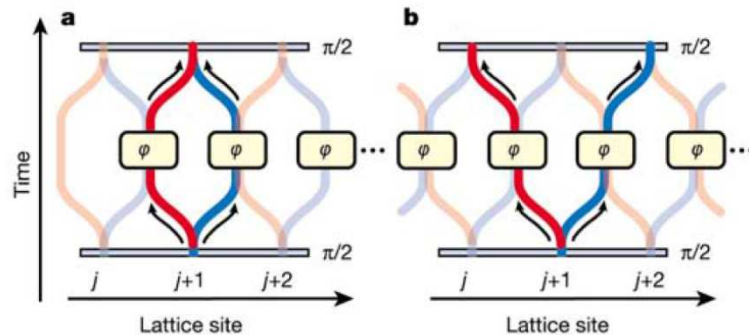
(2) Each bridge simulates two-qubit gate (CNOT)



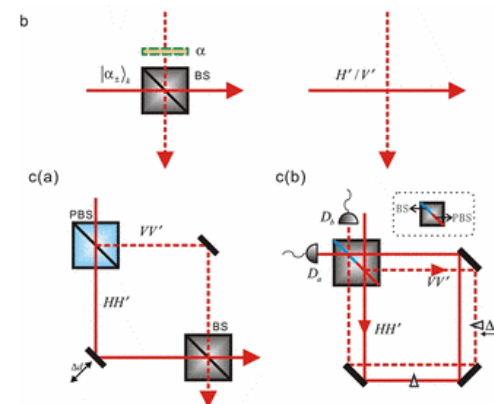
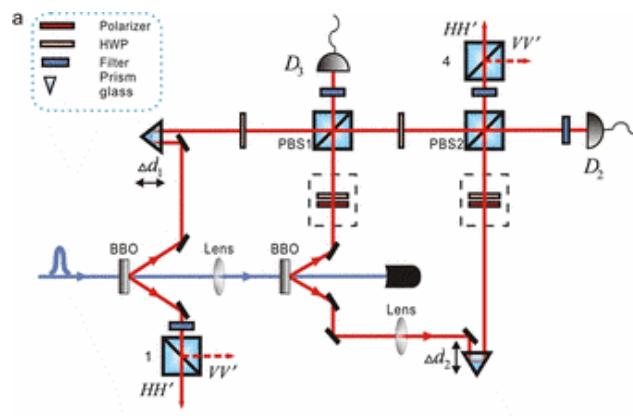
2D or higher dimension is needed for universal QC &
Graph connectivity is essential (percolation)

Realizations of cluster states

- Bloch's group: controlled collision in cold atoms (Nature 2003)



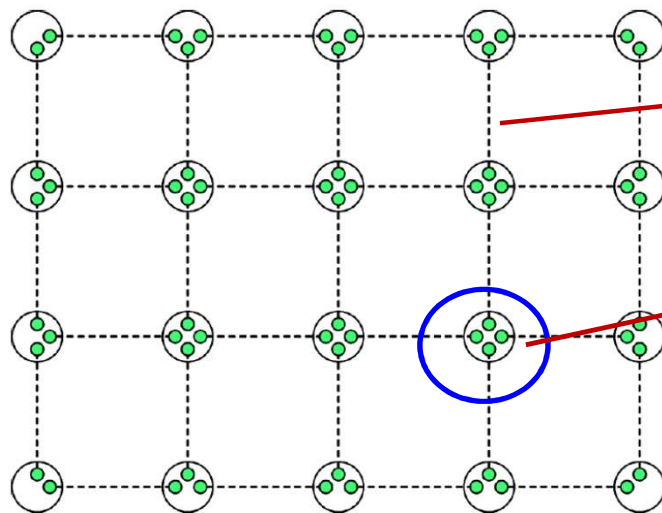
- J-W Pan's group: 4-photon 6 qubit and CNOT (PRL 2010)



Cluster state: a valence-bond picture

- Cluster state = a valence-bond state
= a projected entangled pair state (PEPS)

[Verstraete & Cirac '04]



➤ Bond of two virtual qubits =
 $CZ|++\rangle = |0\rangle|+\rangle + |1\rangle|-\rangle$

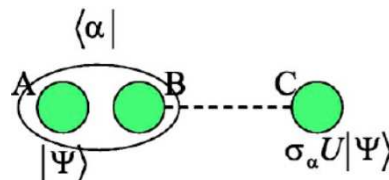
➤ Projection of several virtual qubits to physical qubit =

$$P = |0\rangle\langle 0000| + |1\rangle\langle 1111|$$

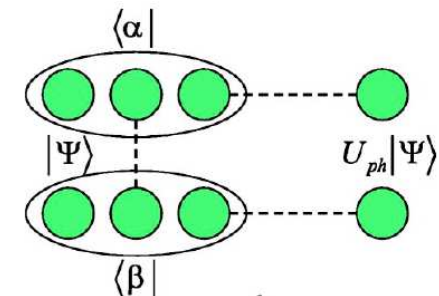
- Quantum computation via teleportation

[see also Gottesman & Chuang '99]

➤ 1-qubit gate:



➤ 2-qubit gate:

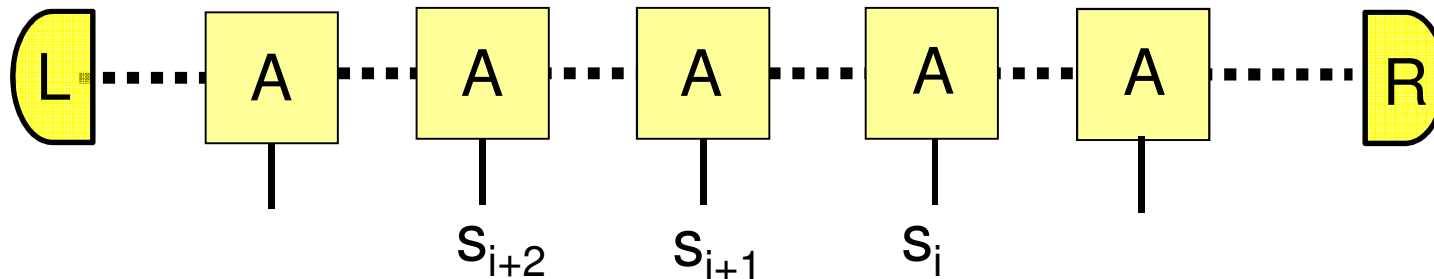


QC in correlation space

- Previous picture of valence bond was generalized by Gross and Eisert using matrix product states (MPS) and PEPS

[Gross & Eisert '07,
Gross et al. '07]

- Illustrate with 1D cluster state:

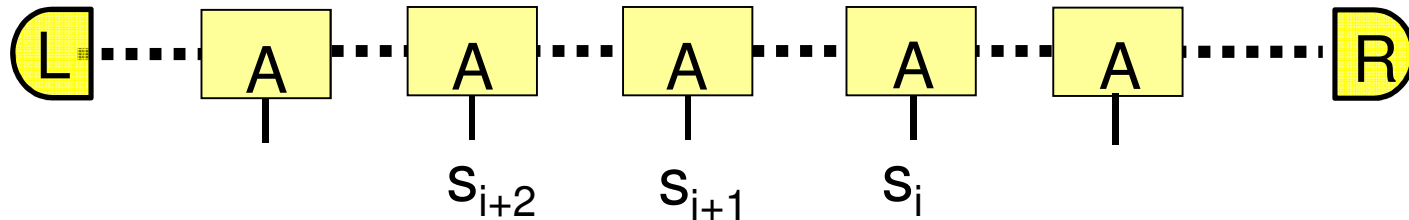


$$|\Psi\rangle = \sum_{\{s\}'s} \vec{L} \cdot A_{s_n} \cdots A_{s_{i+1}} \cdot A_{s_i} \cdots A_{s_1} \cdot \vec{R} |s_n, \dots, s_i, \dots s_1\rangle$$

- Measurement outcome ϕ_i at site i : $A(\phi_i) \equiv \sum_{s_i} \langle \phi_i | s_i \rangle A_{s_i}$

$$\langle \phi_n, \dots \phi_i, \dots, \phi_1 | \Psi \rangle = \vec{L} \cdot A(\phi_n) \cdots A(\phi_i) \cdots A(\phi_1) \cdot \vec{R}$$

Cluster state QC: in correlation space



□ Measurement outcome ϕ_i at site i : $A(\phi_i) \equiv \sum_{s_i} \langle \phi_i | s_i \rangle A_{s_i}$

$$\langle \phi_n, \dots, \phi_i, \dots, \phi_1 | \Psi \rangle = \vec{L} \cdot A(\phi_n) \cdots A(\phi_i) \cdots A(\phi_1) \cdot \vec{R}$$

□ As spins are measured, the boundary vector R is operated by gates

$$|R\rangle \rightarrow A_1(\phi_1)|R\rangle \rightarrow A_1(\phi_2)A_1(\phi_1)|R\rangle \rightarrow \dots$$

□ For 1D cluster state: $A(0) = |+\rangle\langle 0|$, $A(1) = |-\rangle\langle 1|$

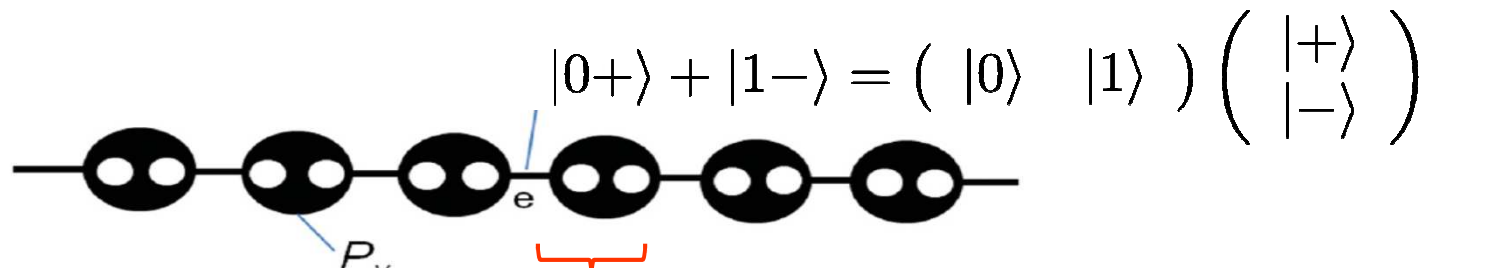
➔ measure in basis $|\pm \xi\rangle = (e^{-i\xi/2}|0\rangle \pm e^{i\xi/2}|1\rangle)/\sqrt{2}$

➔ obtain same 1-qubit gate as before:

$$A(\xi, s) = e^{i\xi/2}|+\rangle\langle 0| + (-1)^s e^{-i\xi/2}|-\rangle\langle 1| = H e^{i\xi Z/2} Z^s$$

□ 2-qubit gates use 2D PEPS ➔ see Gross & Eisert '07

Comment: deriving MPS for cluster state



$$|0+\rangle + |1-\rangle = \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} = \begin{pmatrix} | + 0\rangle & | + 1\rangle \\ | - 0\rangle & | - 1\rangle \end{pmatrix}$$

$$P_v = |0\rangle\langle 00| + |1\rangle\langle 11|$$

□ MPS form:

$$P_v \begin{pmatrix} | + 0\rangle & | + 1\rangle \\ | - 0\rangle & | - 1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |0\rangle & |1\rangle \\ |0\rangle & -|1\rangle \end{pmatrix} = |0\rangle (|+\rangle\langle 0|) + |1\rangle (|-\rangle\langle 1|)$$

$$A(0) = |+\rangle\langle 0|, \quad A(1) = |-\rangle\langle 1|$$

Related talk:

Monday Session A: 4. [3:00-3:20] **Anurag Anshu, Itai Arad and Aditya Jain.** *How local is the information in MPS/PEPS tensor networks?*

Cluster states: not unique ground state of 2-body Hamiltonians

- ❑ First proved by Nielsen

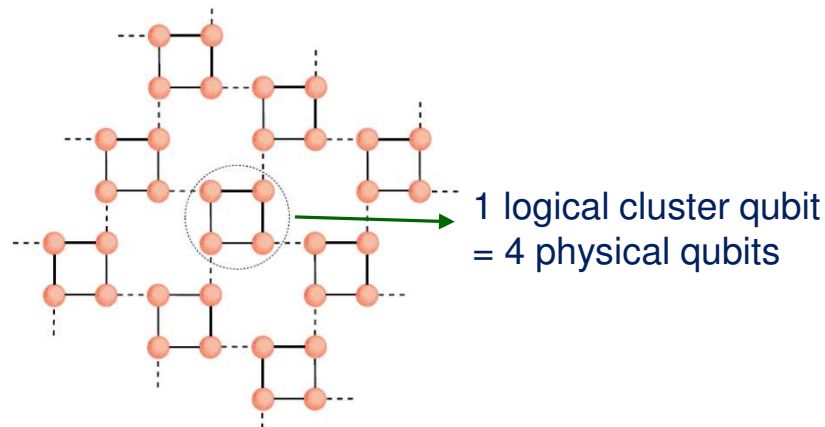
[Haselgrove, Nielsen & Osborne '03, Nielsen '04]

- ❑ Van den Nest et al. proved for general (connected) graph states G :

→ For approximation: ground-state of 2-body Hamiltonian can be ϵ -close to G , but the gap is proportional to ϵ [Van den Nest et al. '08]

- ❑ Bartlett & Rudolph constructed a two-body Hamiltonian such that the ground state is approximately an encoded cluster state

[Bartlett & Rudolph '06]



$$H_S = - \sum_{\mu \in S} \sum_{i \sim i'} \sigma_{(\mu,i)}^z \otimes \sigma_{(\mu,i')}^z$$

$$V = - \sum_{(\mu,i) \sim (v,j)} (\sigma_{(\mu,i)}^z \otimes \sigma_{(v,j)}^x + \sigma_{(\mu,i)}^x \otimes \sigma_{(v,j)}^z)$$

- ❑ Darmawan & Bartlett constructed encoded cluster state by deforming the AKLT Hamiltonian

[Darmawan & Bartlett '14]

Linear optic QC & cluster state

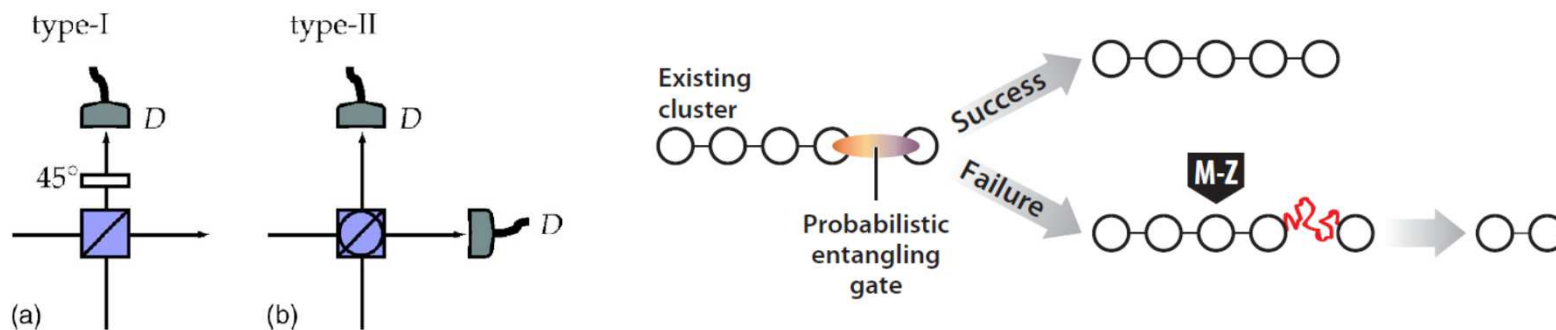
- Linear optic universal QC possible with single photon source, linear optic elements (beam splitters, mirrors, etc) & photon counting

→ High overhead in entangling gates [Knill, Laflamme & Milburn '01]

- Cluster state helps reduce this overhead

→ Grow cluster states efficiently

[Yoran & Reznik '03; Nielsen '04; Browne & Rudolph '05; Kieling, Rudolph & Eisert '07]

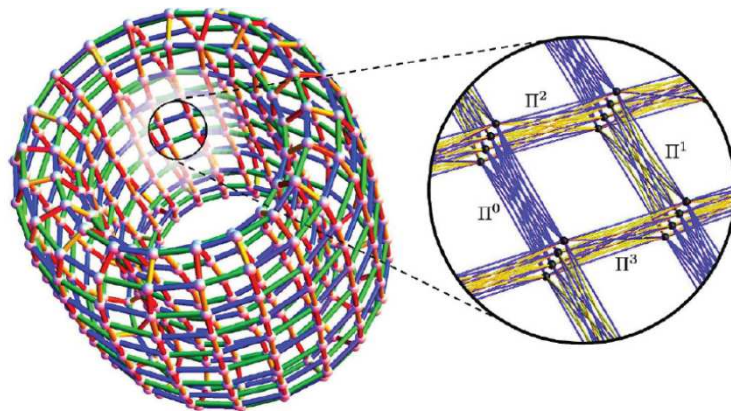


- Experiments: see e.g. [O'Brien Science '07]

Create continuous-variable cluster states

- Use frequency comb and parametric amplifier in cavity

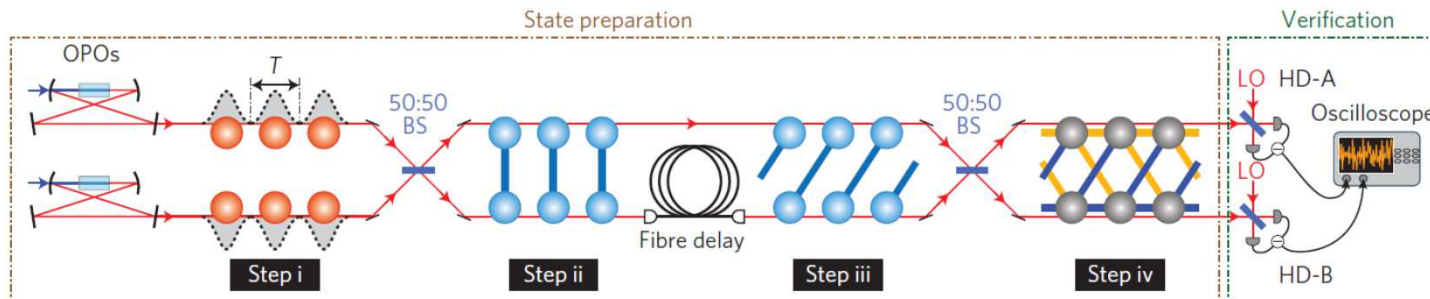
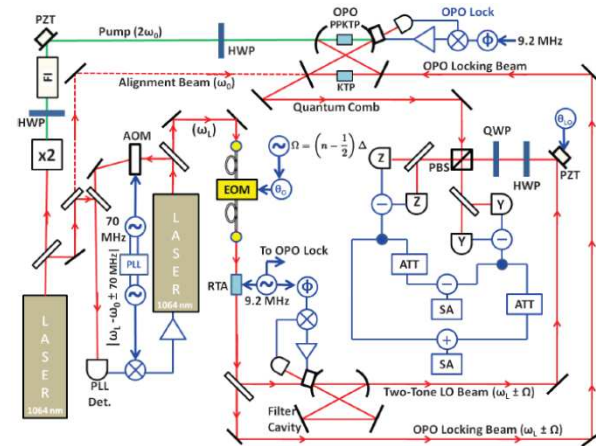
➤ Theory: [Menicucci et al '06, '08]



$$X(s) = e^{-is\hat{p}} \text{ and } Z(t) = e^{it\hat{q}} \quad C_Z = \exp(i\hat{q} \otimes \hat{q})$$

➤ Experiment II: [$> 10,000$ modes in Furusawa group '12]

➤ Experiment I: [60 modes in Pfister group '11]



Related talks:

Thursday Session A:

5. [3:20-3:40] **Hoi-Kwan Lau and Martin**

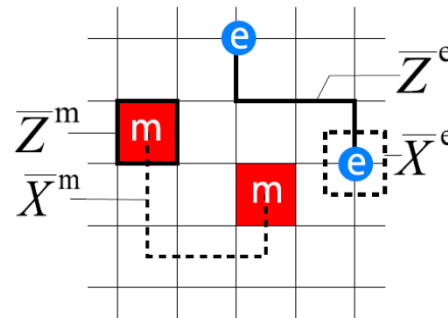
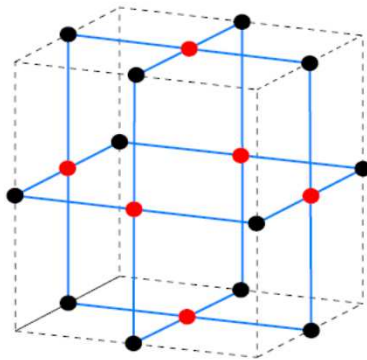
Plenio. *Universal Quantum Computing with Arbitrary Continuous-Variable Encoding*

6. [3:40-4:00] **Alessandro Ferraro, Oussama Houhou, Darren Moore, Mauro Paternostro and Tommaso**

Tufarelli. *Measurement-based quantum computation with mechanical oscillators*

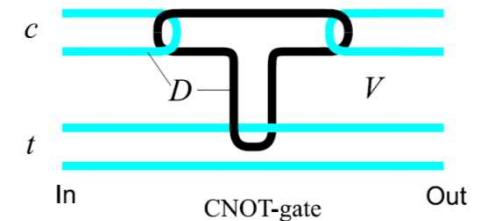
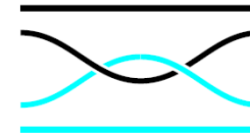
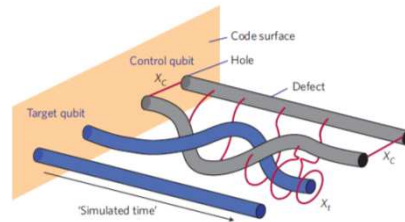
Fault tolerant cluster-state QC

- ❑ Uses a 3d cluster state and implements surface codes in each 2d layer

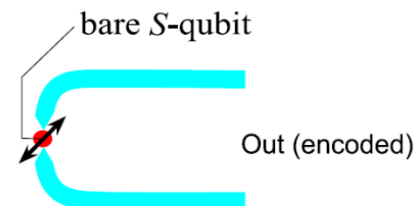


[Raussendorf, Harrington & Goyal '07]

- ❑ CNOT is achievable



- ❑ Uses magic-state distillation to achieve non-Clifford gate



➔ Error threshold 0.75%, qubit loss threshold 24.9%

[Barrett & Stace '10]

Related talk:

Friday 10:30-11:00 [Long] **Guillaume Dauphinais
and David Poulin.** *Fault Tolerant Quantum Memory
for non-Abelian Anyons*

Universal blind quantum computation

[Broadbent, Fitzsimons & Kashefi '09]

□ Using the following cluster state (called brickwork state)

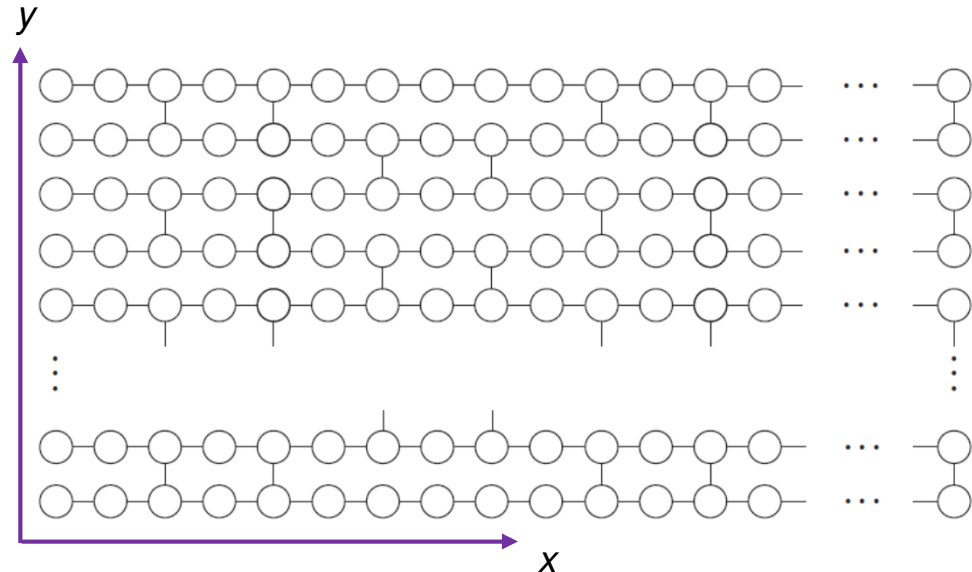
➤ Alice prepares

$$|\Psi\rangle = \bigotimes_{x,y} (|0\rangle_{x,y} + e^{i\theta_{x,y}} |1\rangle_{x,y})$$

with random

$$\theta_{x,y} = 0, \pi/4, \dots, 7\pi/4$$

➤ Bob entangles all qubits according to the brickwork graph via CZ gates



➤ Alice tells Bob what measurement basis for Bob to perform and he returns the outcome (compute like one-way computer)

➔ Alice can achieve her quantum computation without Bob knowing what she computed!!

- 1 Alice computes $\phi'_{x,y}$ where $s_{0,y}^X = s_{0,y}^Z = 0$. $\phi'_{x,y} = (-1)^{s_{x,y}^X} \phi_{x,y} + s_{x,y}^Z \pi$
- 2 Alice chooses $r_{x,y} \in_R \{0, 1\}$ and computes $\delta_{x,y} = \phi'_{x,y} + \theta_{x,y} + \pi r_{x,y}$.
- 3 Alice transmits $\delta_{x,y}$ to Bob. Bob measures in the basis $\{|+\delta_{x,y}\rangle, |-\delta_{x,y}\rangle\}$.
- 4 Bob transmits the result $s_{x,y} \in \{0, 1\}$ to Alice.
- 5 If $r_{x,y} = 1$ above, Alice flips $s_{x,y}$; otherwise she does nothing.

➔ Realized in an exp.
Barz et al. 2012

We have seen the cluster states on the square lattice and the brickwork lattice for universal for quantum computation

Q: How much do we know about the general cluster/graph states?

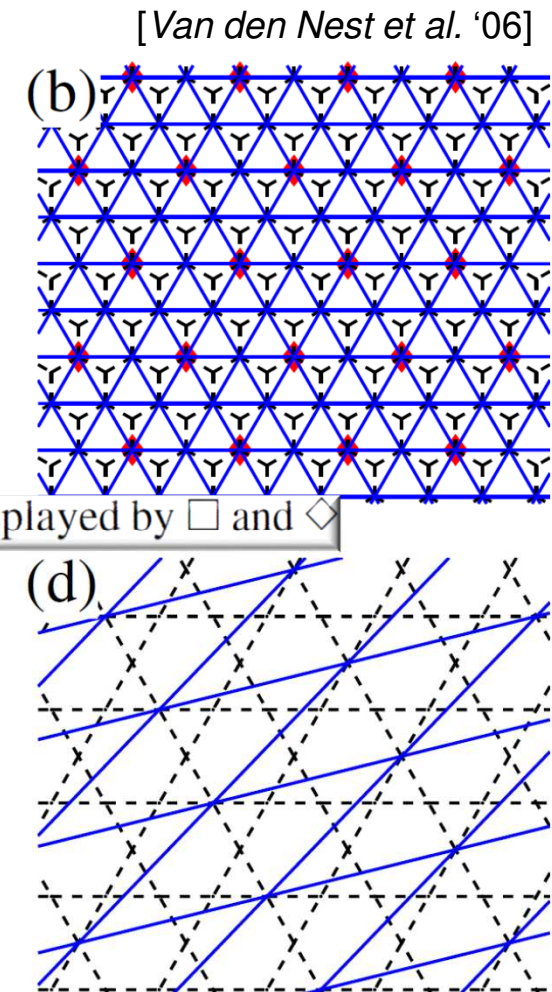
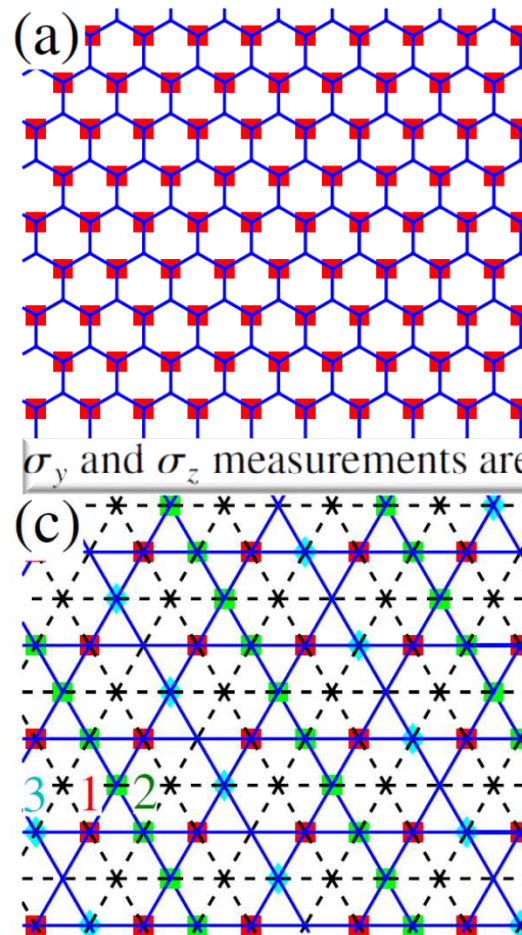
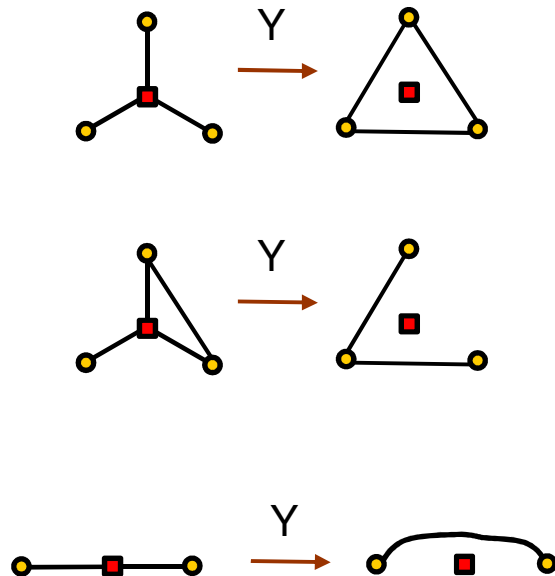
Universality in graph/cluster states

- ❖ Beyond square & brickwork: other *2D graph/cluster states* on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal
 - ➔ Can use local measurement to convert one to the other (with fewer qubits, but still macroscopic) [Van den Nest et al. '06]

Graph states on regular lattices

- ❖ Beyond square & brickwork: other *2D graph/cluster states* on regular lattices, e.g. triangular, honeycomb, kagome, etc. are universal

→ local measurement converts one to another



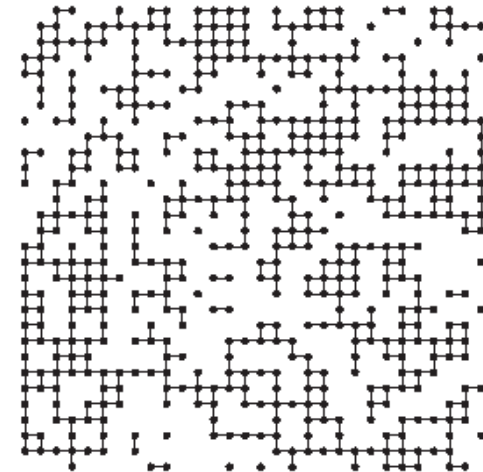
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(with fewer qubits, but still macroscopic)

- ❖ Faulty square lattice (degree ≤ 4)

[Browne et al. '08]

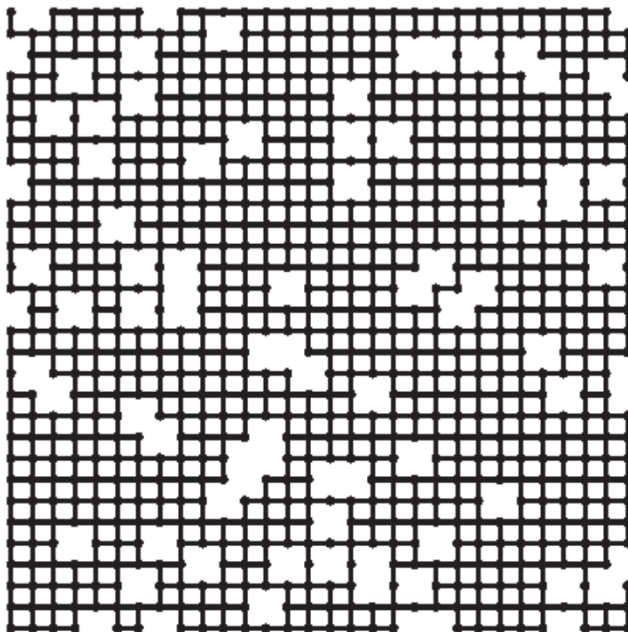
- As long as it is sufficiently connected (a la percolation), can find sub-graph \sim honeycomb




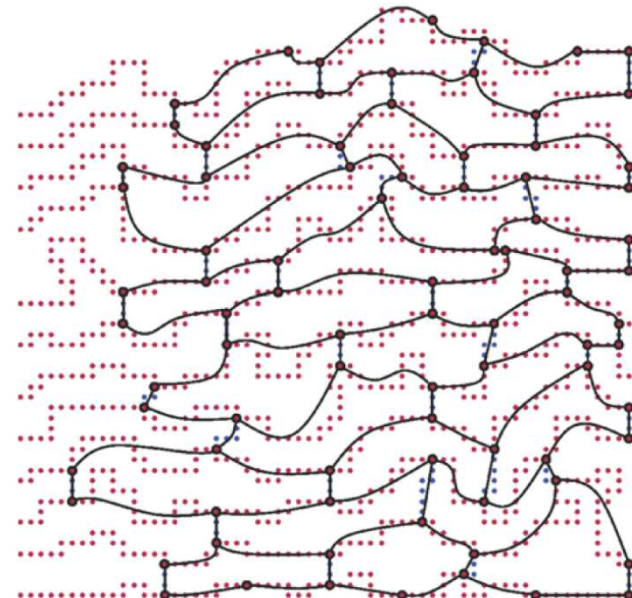
Cluster state on faulty lattice

[Browne et al. '08]

- ❖ No qubits on empty sites (degree ≤ 4)
 \longleftrightarrow site percolation
- ❖ But assume perfect CZ gates $|G\rangle = \bigotimes_{\langle i,j \rangle} CZ_{ij} (|+\rangle|+\rangle \cdots |+\rangle)$
- ❖ As long as probability of occupied sites $>$ site percolation threshold
 \rightarrow still universal for MBQC



Local
measurement




Universality in graph/cluster states

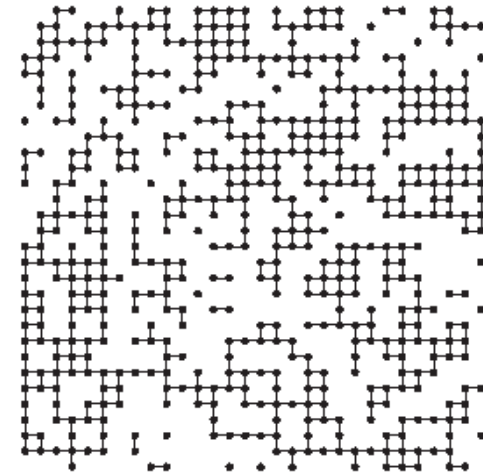
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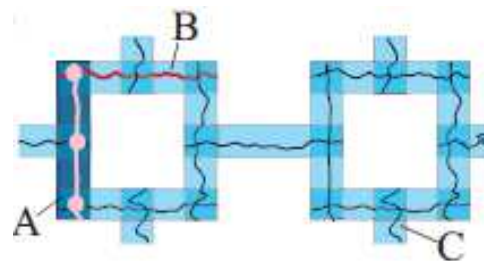
[Browne et al. '08]

→ As long as it is sufficiently connected
(a la percolation), can find sub-graph \sim honeycomb



- ❖ Any 2D planar random graphs in supercritical phase of percolation are universal

[Wei, Affleck & Raussendorf. '12]



Other universal states

- ❑ So far no complete characterization for resource states

- ❑ Can they be unique ground state with 2-body Hamiltonians with a finite gap?

➔ If so, create resources by cooling!

- ❖ TriCluster state [Chen et al. '09]

- ❖ Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [AKLT '87, '88]

- 1D (not universal): [Gross & Eisert '07, '10] [Brennen & Miyake '08?]

- 2D (universal): [Wei, Affleck & Raussendorf '11] [Miyake '11] [Wei et al. '13-'15]

- ❖ Symmetry-protected topological states

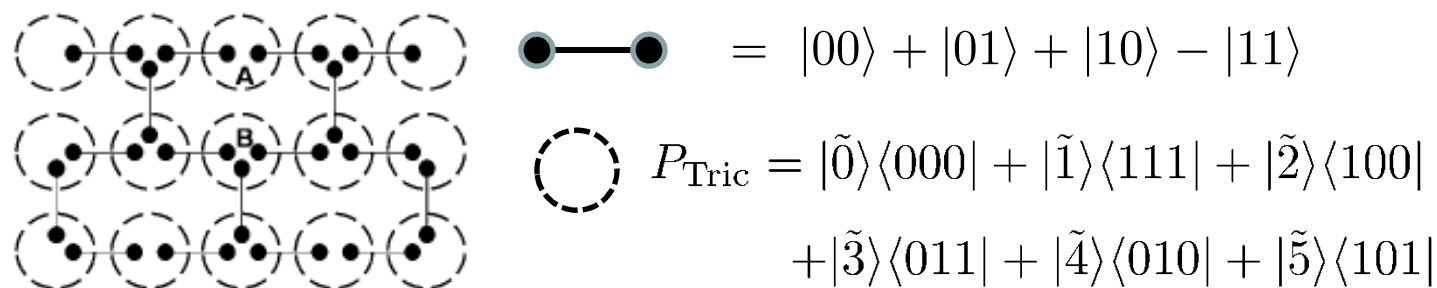
- 1D (not universal): [Else, Doherty & Bartlett '12] [Miller & Miyake '15] [Prakash & Wei '15]

- 2D (universal, *but not much explored*): [Poulsen Nautrup & Wei '15] [Miller & Miyake '15]



Example ground state of two-body Hamiltonian as computational resource

□ TriCluster state (6-level) [Chen, Zeng, Gu, Yoshida & Chuang, PRL'09]



$$H_{\text{triC}}^* = \sum_a (h_{ab} + h_{ba} + h_a)$$

$$\begin{aligned}
 h_{ab} = & 2(2S_{a_z} - 5)(2S_{a_z} - 3)(2S_{a_z} - 1)(2S_{a_z} + 1)(4S_{a_z} + 11) \\
 & (2S_{b_z} + 5)(2S_{b_z} + 3)(2S_{b_z} - 1)(2S_{b_z} + 1)(4S_{b_z} - 11) \\
 & - 75\sqrt{2}S_{a_+}(2S_{a_z} - 5)(2S_{a_z} + 3)(2S_{a_z} - 1)(2S_{a_z} + 1) \\
 & (48S_{b_z}^4 + 64S_{b_z}^3 - 280S_{b_z}^2 - 272S_{b_z} + 67) \\
 & + 75\sqrt{2}(48S_{a_z}^4 - 64S_{a_z}^3 - 280S_{a_z}^2 + 272S_{a_z} + 67) \\
 & S_{b_+}(2S_{b_z} - 5)(2S_{b_z} - 3)(2S_{b_z} - 1)(2S_{b_z} + 3) \\
 & + 4\sqrt{10}S_{a_+}^3(2S_{a_z} - 1)(2S_{a_z} - 3) \times \\
 & (128S_{b_z}^5 + 560S_{b_z}^4 - 2840S_{b_z}^2 - 3848S_{b_z} + 675) \\
 & + 4\sqrt{10}(128S_{a_z}^5 - 560S_{a_z}^4 + 2840S_{a_z}^2 - 3848S_{a_z} - 675) \\
 & S_{b_+}^3(2S_{b_z} - 5)(2S_{b_z} - 3) + h.c.
 \end{aligned}$$

$$\begin{aligned}
 h_a = & -25(2S_{a_z} - 5)(2S_{a_z} - 3)(2S_{a_z} + 3)(2S_{a_z} + 5) \\
 & + 25S_{a_+}^3(2S_{a_z} - 5)(2S_{a_z} - 1) \\
 & (224S_{b_z}^5 - 16S_{b_z}^4 - 1968S_{b_z}^3 + 40S_{b_z}^2 + 3550S_{b_z} - 9) \\
 & - 12S_{a_+}^5 \\
 & (416S_{b_z}^5 - 80S_{b_z}^4 - 3600S_{b_z}^3 + 520S_{b_z}^2 + 5994S_{b_z} - 125) \\
 & + h.c. + (a \leftrightarrow b),
 \end{aligned}$$

Too much entanglement is useless

- States (n -qubit) possessing too much geometric entanglement E_g are not universal for QC (i.e if $E_g > n - \delta$)

[Gross, Flammia & Eisert '09;
Bremner, Mora & Winter '09]

$$E_g(|\Psi\rangle) = -\log_2 \max_{\phi \in \mathcal{P}} |\langle \phi | \Psi \rangle|^2 \quad \mathcal{P} = \text{set of product states}$$

- Intuition: if state is very high in geometric entanglement, every local measurement outcome has low probability
 - whatever local measurement strategy, the distribution of outcomes is so random that one can simulate it with a random coin (thus not more powerful than classical random string)
- Moreover, states with high entanglement are typical:
 - those with $E_g < n - 2\log_2(n) - 3$ is rare, i.e. with fraction $< e^{-n^2}$
 - Universal resource states are rare!!

Outline

I. Introduction

II. One-way (measurement-based) quantum computers

III. Other entangled resource states: AKLT family

IV. Summary

A new direction: valence-bond ground states of isotropic antiferromagnet

□ AKLT (Affleck-Kennedy-Lieb-Tasaki) states/models

- ❖ Importance: provide strong support for Haldane's conjecture on spectral properties of spin chains [AKLT '87,88]
- ❖ Provide concrete example for symmetry-protected topological order [Gu & Wen '09, '11]

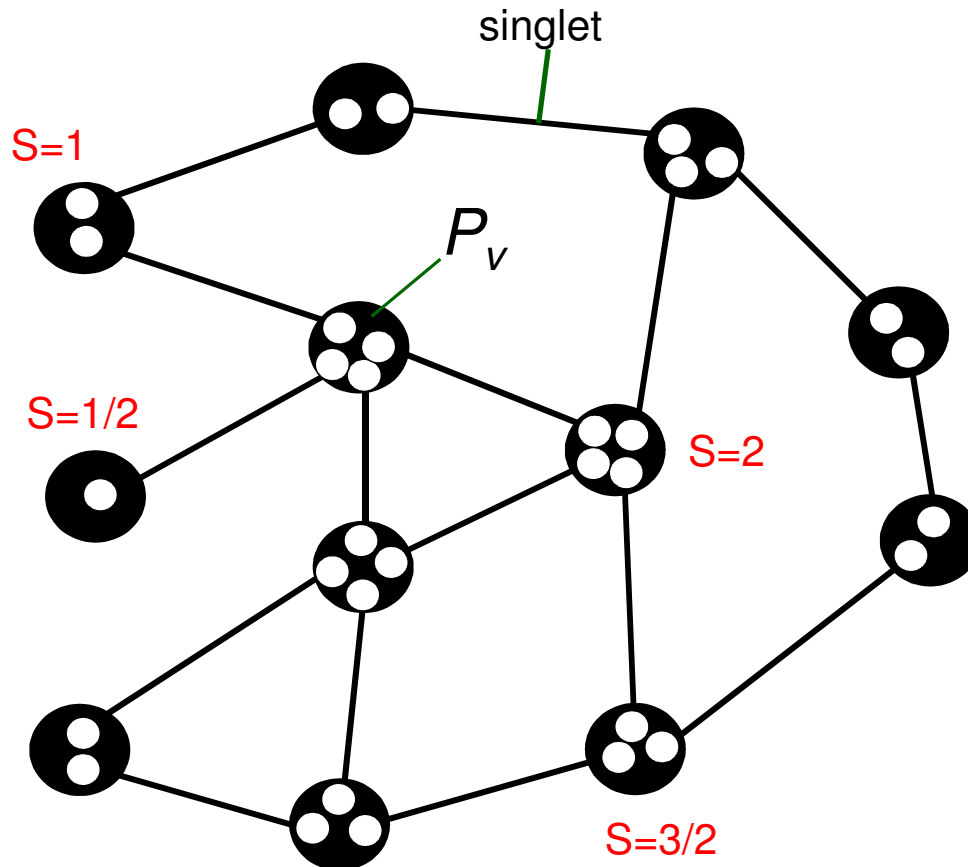
□ States of spin $S=1, 3/2, 2, \dots$ (defined on any lattice/graph)

➔ Unique* ground states of gapped# two-body isotropic Hamiltonians

$$H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j) \quad f(x) \text{ is a polynomial}$$

*w/ appropriate boundary conditions; #gap proved in 1D; evidence in 2D: *Garcia-Saez, Murg, Wei* '12

(hybrid) AKLT state defined on any graph



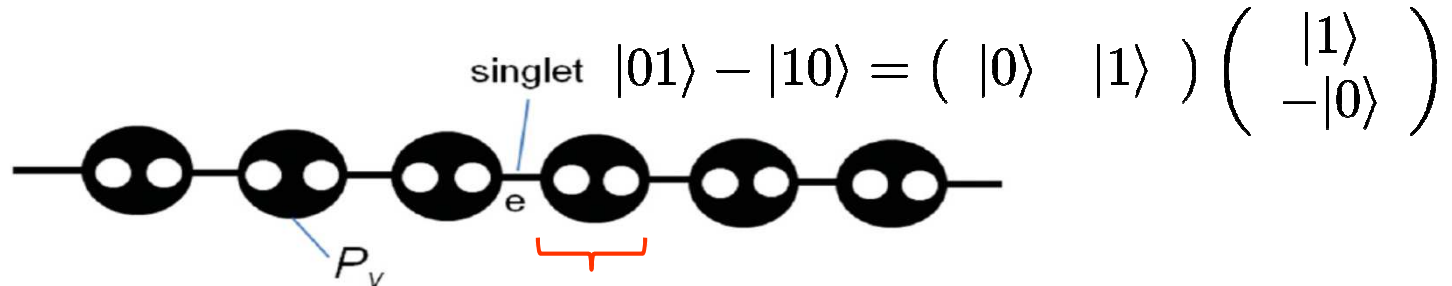
- # virtual qubits = # neighbors
- $S = \# \text{ neighbors} / 2$
- Physical spin Hilbert space = symmetric subspace of qubits

P_v = projection to symmetric subspace of n qubit \equiv spin $n/2$

1D AKLT state for simulating 1-qubit gates

- Easy to see from its matrix product state (MPS)

[Gross & Eisert, PRL '07][Brennen & Miyake, PRL '09]



$$\begin{pmatrix} |1\rangle \\ -|0\rangle \end{pmatrix} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} = \begin{pmatrix} |10\rangle & |11\rangle \\ -|00\rangle & -|01\rangle \end{pmatrix}$$

$$P_v = | + 1 \rangle \langle 00| + |0\rangle (\langle 01| + \langle 10|) / \sqrt{2} + | - 1 \rangle \langle 11|$$

- MPS form: $|0\rangle \equiv |z\rangle, | + 1 \rangle \equiv -(|x\rangle + i|y\rangle) / \sqrt{2}, | - 1 \rangle \equiv (|x\rangle - i|y\rangle) / \sqrt{2}$

$$P_v \begin{pmatrix} |10\rangle & |11\rangle \\ -|00\rangle & -|01\rangle \end{pmatrix} = \begin{pmatrix} |0\rangle / \sqrt{2} & | - 1 \rangle \\ -| + 1 \rangle & -|0\rangle / \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} (|x\rangle X + |y\rangle Y + |z\rangle Z)$$

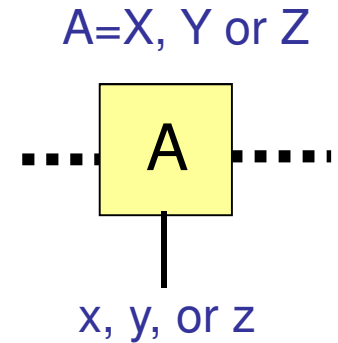
→ Gates with superposition of X, Y, Z are achievable

→ Arbitrary 1-qubit gates possible (but universal QC requires 2-qubit gates) → any 2D AKLT states universal?

Hamiltonian & SPT order

- 1D spin-1 AKLT state $|x\rangle X + |y\rangle Y + |z\rangle Z$ is ground state of the gapped 2-body Hamiltonian

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2$$



- AKLT is a symmetry-protected topological (SPT) state, e.g. by $Z_2 \times Z_2$ symmetry (rotation around x or z by 180°)
- Under transformation on physical spins:

$$|0\rangle \equiv |z\rangle, \quad | + 1 \rangle \equiv -(|x\rangle + i|y\rangle)/\sqrt{2}, \quad | - 1 \rangle \equiv (|x\rangle - i|y\rangle)/\sqrt{2}$$

$$U_z(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{array}{l} |z\rangle \rightarrow |z\rangle, \quad |x\rangle \rightarrow -|x\rangle, \quad |y\rangle \rightarrow -|y\rangle \\ A \rightarrow Z \cdot A \cdot Z \end{array}$$

$$U_x(\pi) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} |z\rangle \rightarrow -|z\rangle, \quad |x\rangle \rightarrow |x\rangle, \quad |y\rangle \rightarrow -|y\rangle \\ A \rightarrow X \cdot A \cdot X \end{array}$$

→ Projective representation (e.g. Z & X) of symmetry implies SPT order

SPT order of cluster state

- MPS for cluster state (single site):

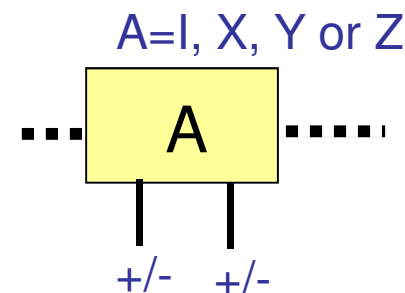
$$A(0) = |+\rangle\langle 0|, \quad A(1) = |-\rangle\langle 1|$$

$$\rightarrow \text{+/- basis: } A(+) \sim A(0) + A(1) = H, \quad A(-) = HZ$$

- Two sites:

$$A(++) = H^2 = 1, \quad A(+-) = H(HZ) = Z$$

$$A(-+) = (HZ)H = X, \quad A(--) = (HZ)^2 = XZ$$



- Under XIXI... on physical spins:

$$\left. \begin{array}{l} A(++) \rightarrow A(++) \\ A(+-) \rightarrow A(+-) \\ A(-+) \rightarrow -A(-+) \\ A(--) \rightarrow -A(--) \end{array} \right\} A(\alpha, \beta) \rightarrow Z \cdot A(\alpha, \beta) \cdot Z$$

- Similarly for IXIX... : $A(\alpha, \beta) \rightarrow X \cdot A(\alpha, \beta) \cdot X$

\rightarrow projective representation \rightarrow SPT order

SPT order & gates

- AKLT is a symmetry-protected topological (SPT) state, e.g. by $Z_2 \times Z_2$ symmetry (rotation around x or z by 180°) with Hamiltonian

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2$$

- 1D cluster state is also a SPT state, e.g. by $Z_2 \times Z_2$ symmetry ($XIXI\dots$ or $IXIX\dots$) with Hamiltonian

$$H = - \sum_i Z_{i-1} X_i Z_{i+1}$$

- Generic states in such 1D SPT phase

$$A_\alpha = \sigma_\alpha \otimes B_\alpha$$

logical
subspace junk
subspace

[Else et al. '12]

[Prakash & Wei '15]

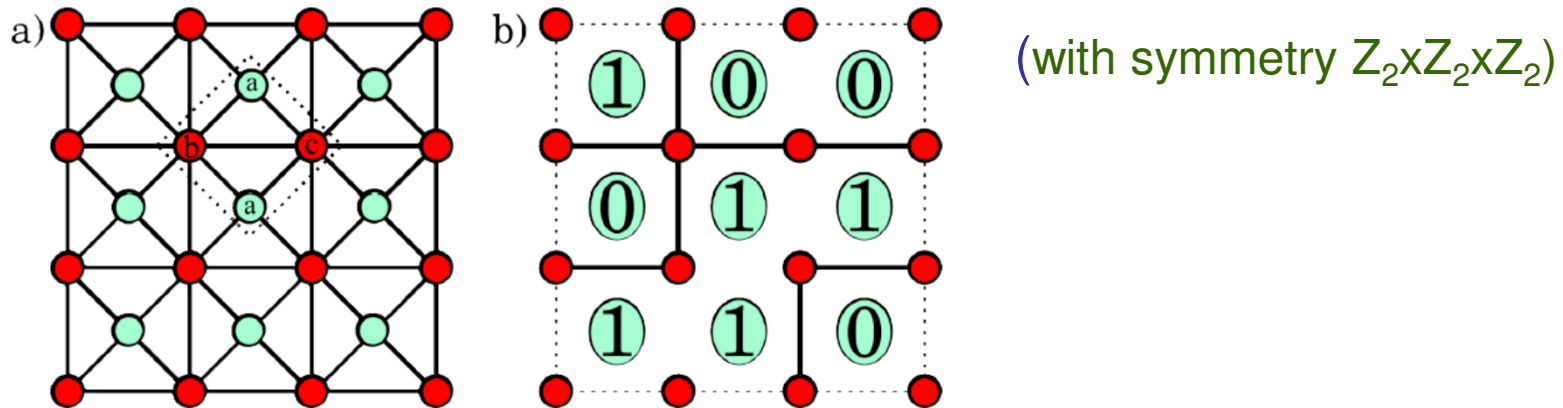
→ Only identity gate (up to Pauli) is protected

→ But arbitrary 1-qubit gate is possible, e.g. with S_4 symmetry [Miller & Miyake '15]

2D SPT states for universal QC

- A “Control-control-Z state”: [Miller & Miyake ‘15]

$\psi =$ CCZ (Control-Control-Z) gates applied to all triangles with $|+++ \dots ++>$

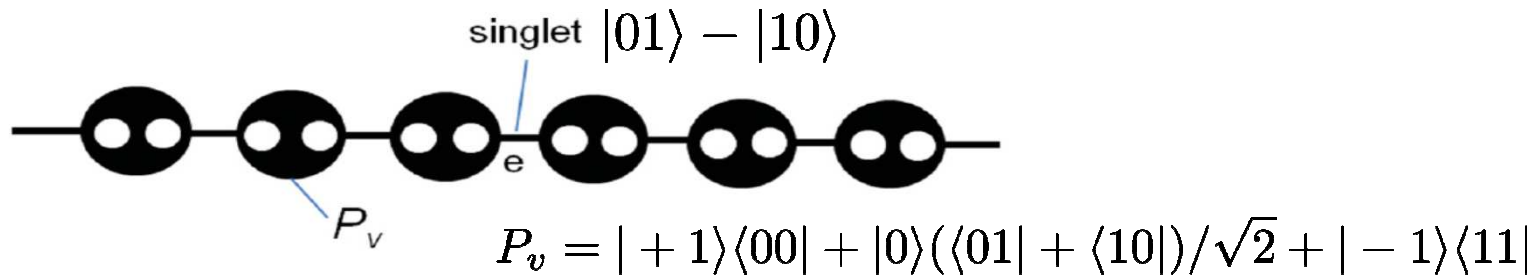


- Fixed-point wavefunctions of generic SPT states (with any nontrivial SPT order) are universal resource; see

Thursday Session A: 4. [3:00-3:20] **Hendrik Poulsen Nautrup and Tzu-Chieh Wei.** *Symmetry-protected topologically ordered states for universal quantum computation*

In the remaining, we will focus on AKLT family of states for universal quantum computation

Converting 1D AKLT state to cluster state



- Via adaptive local measurement (i.e. state reduction)

[Chen, Duan, Ji & Zeng '10]

- Via fixed POVM [Wei, Affleck & Raussendorf '11]

→ generalizable to 2D AKLT: $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$

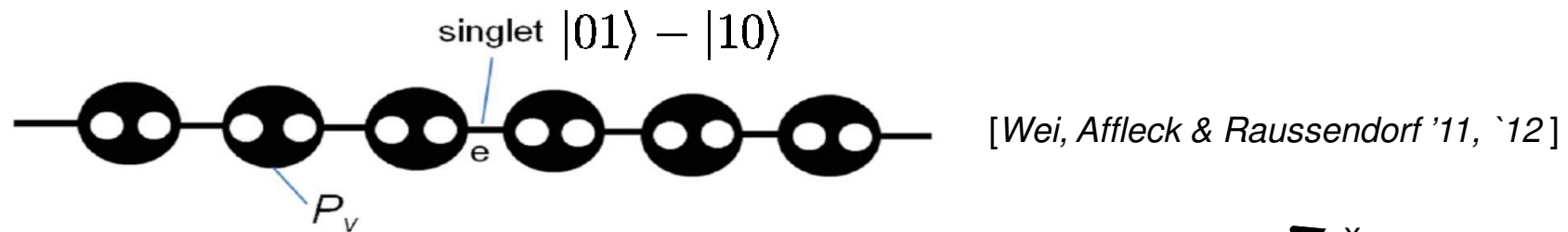
$$F_x \sim |S_x = 1\rangle\langle S_x = 1| + |S_x = -1\rangle\langle S_x = -1| \sim |++\rangle\langle ++| + |--\rangle\langle --|$$

$$F_x \sim |S_y = 1\rangle\langle S_y = 1| + |S_y = -1\rangle\langle S_y = -1| \sim |i, i\rangle\langle i, i| + |-i, -i\rangle\langle -i, -i|$$

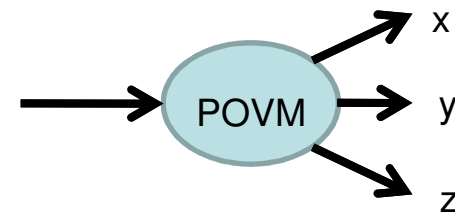
$$F_z \sim |S_z = 1\rangle\langle S_z = 1| + |S_z = -1\rangle\langle S_z = -1| \sim |00\rangle\langle 00| + |11\rangle\langle 11|$$

→ Outcome labeled by x, y, z: $|\psi\rangle \rightarrow F_\alpha |\psi\rangle$

POVM: 1D AKLT state \rightarrow cluster state



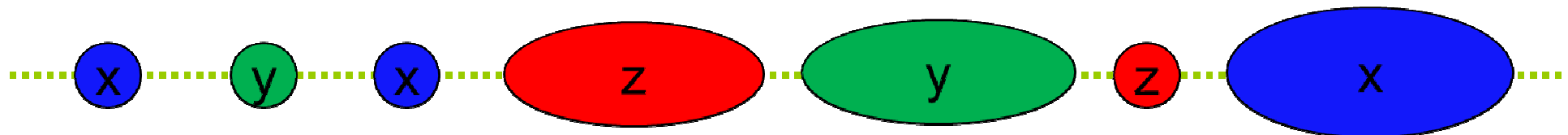
□ POVM: $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$



e.g. for the outcome (labeled x, y, z)



\rightarrow the post-measurement state is an encoded 1D cluster state with graph:

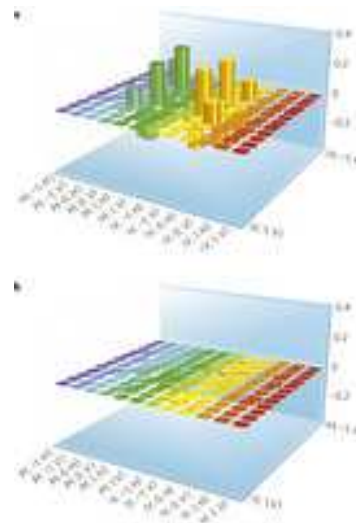
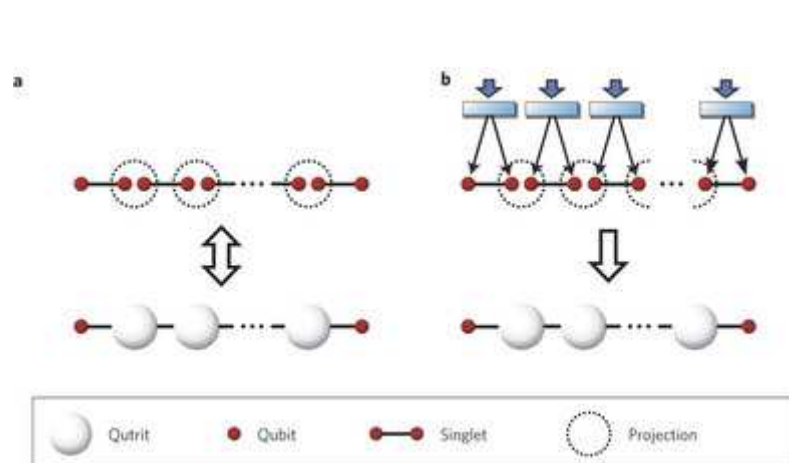


\rightarrow 1 logical qubit = 1 domain = consecutive sites with same outcome

\rightarrow This generalizes to some 2D AKLT states (with $S \leq 2$)

Realizations of 1D AKLT state

- Resch's group: photonic implementation (Nature Phys 2011)

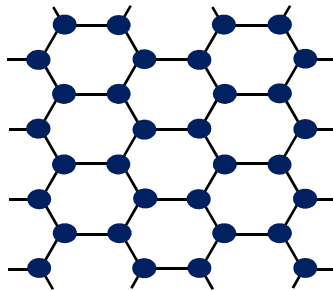


2D AKLT states for quantum computation?

□ On various lattices

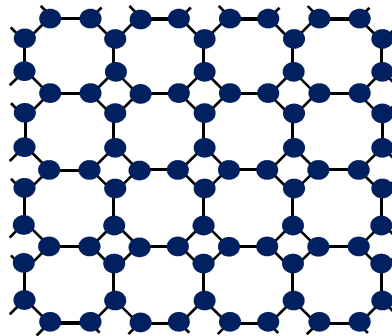
*Wei, Affleck & Raussendorf, PRL '11; Miyake '11;
Wei, PRA '13, Wei, Haghnegahdar & Raussendorf, PRA '14
Wei & Raussendorf '15*

😊 honeycomb

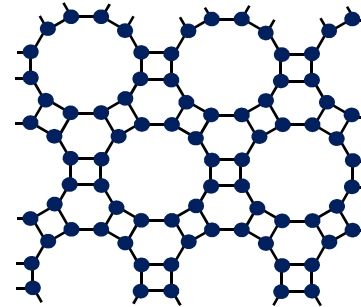


spin-3/2:

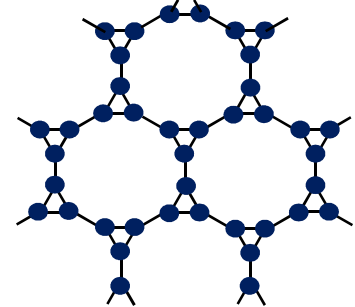
😊 square-octagon



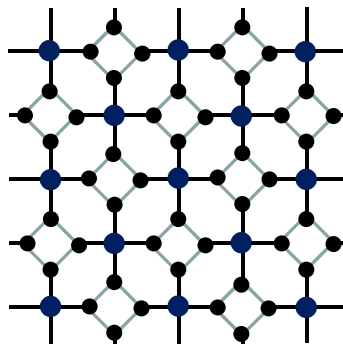
😊 'cross'



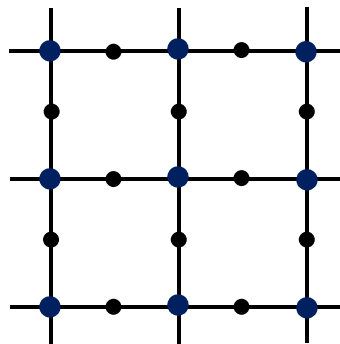
😞 star



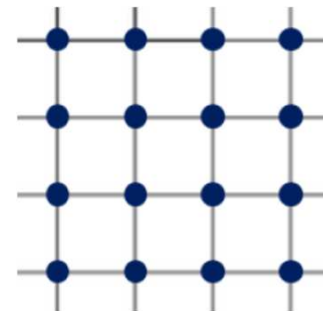
😊 square-hexagon
(spin-2 spin-3/2 mixture)



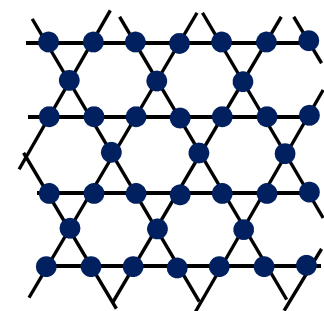
😊 decorated-square
(spin-2 spin-1 mixture)



😊 square
(spin-2)

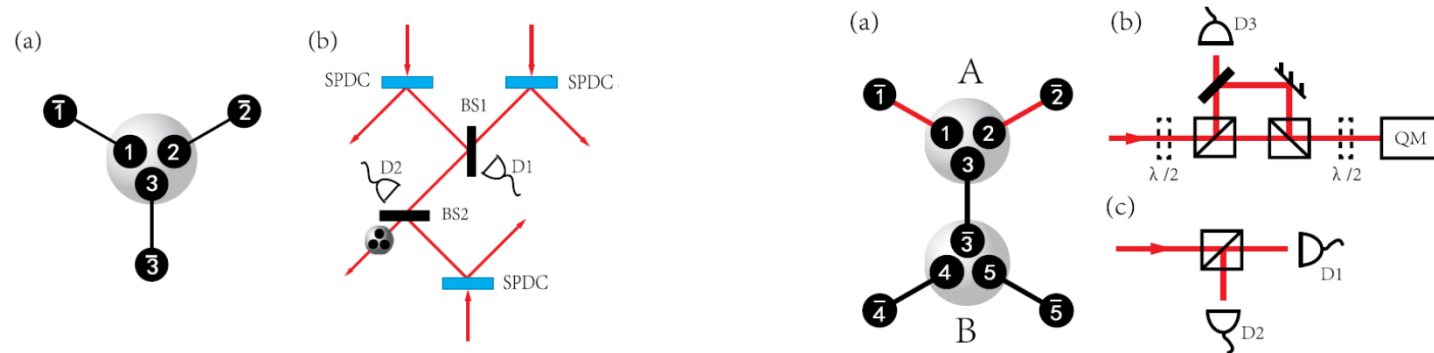


😞 Kagome
(spin-2)



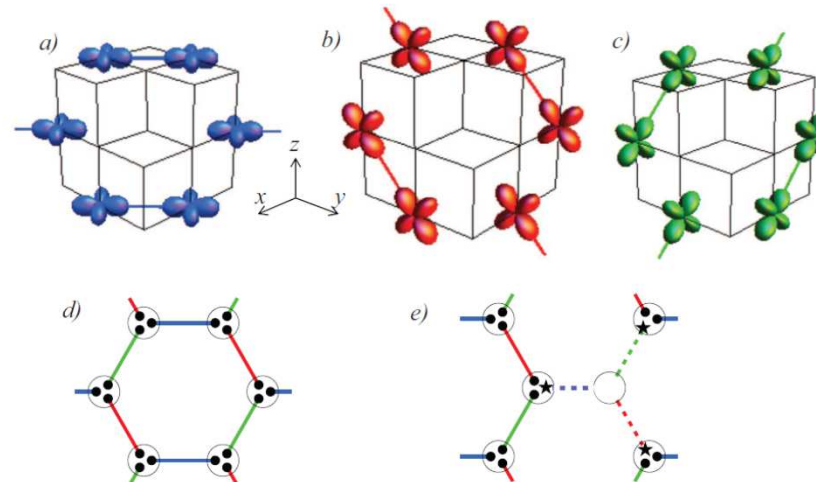
Proposal for 2D AKLT states

- Liu, Li and Gu [JOSA B 31, 2689 (2014)]



- Koch-Janusz, Khomskii & Sela [PRL 114, 247204 (2015)]

t_{2g} electrons in Mott insulator



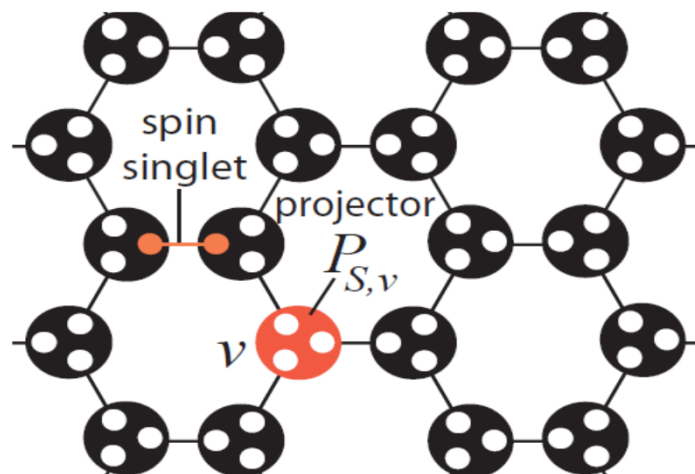
AKLT states on trivalent lattices

- Each site: three virtual qubits $\bullet \equiv \text{spin } 3/2$ (in general: $S = \text{\#nbr} / 2$)

→ physical spin = symmetric subspace of qubits

- Two virtual qubits on an edge form a **singlet** 

$$P = |3/2\rangle\langle 000| + | - 3/2\rangle\langle 111| + |1/2\rangle\langle W| + | - 1/2\rangle\langle \bar{W}|$$



$$|000\rangle \leftrightarrow \left| S = \frac{3}{2}, S_z = \frac{3}{2} \right\rangle$$

$$|111\rangle \leftrightarrow \left| \frac{3}{2}, -\frac{3}{2} \right\rangle$$

→ Effective qubit

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left| \frac{3}{2}, \frac{1}{2} \right\rangle$$

$$|\bar{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left| \frac{3}{2}, -\frac{1}{2} \right\rangle$$

Use generalized measurement (POVM)

$$F_z = \sqrt{\frac{2}{3}} \left(\left| \frac{3}{2} \right\rangle \left\langle \frac{3}{2} \right|_z + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_z \right)$$

[**Wei, Affleck & Raussendorf** '11
Miyake '11]

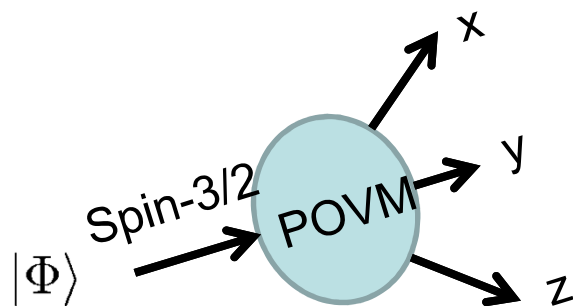
$$F_x = \sqrt{\frac{2}{3}} \left(\left| \frac{3}{2} \right\rangle \left\langle \frac{3}{2} \right|_x + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_x \right)$$

Completeness:

$$F_y = \sqrt{\frac{2}{3}} \left(\left| \frac{3}{2} \right\rangle \left\langle \frac{3}{2} \right|_y + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_y \right)$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$$

- POVM gives random outcome x, y and z at each site



$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z} |\Phi\rangle$$

- Can show POVM on all sites converts AKLT to a graph state
(graph depends on random x, y and z outcomes)

Proving graph state

Let us first explain the notation. Consider a central vertex $C \in V(G_0(\{F\}))$ and all its neighboring vertices $C_\mu \in V(G_0)$. Denote the POVM outcome for all \mathcal{L} sites $v \in C, C_\mu$ by a_c and a_μ , respectively. Denote by E_μ the set of \mathcal{L} edges that run between C and C_μ . Denote by E_c the set of \mathcal{L} edges internal to C . Denote by V_c the set of all qubits in C , and by V_μ the set of all qubits in C_μ . (Recall that there are four qubit locations per \mathcal{L} vertex $v \in C, C_\mu$.) Extending Eq. (33) of Ref. [17] to the spin-2 case, we have

$$\begin{aligned} \mathcal{K}_C &= \bigotimes_{\mu} \bigotimes_{e \in E_\mu} (-1)^{\sigma_{a_\mu}^{(u(e))} \sigma_{a_\mu}^{(v(e))}} \bigotimes_{e' \in E_c} (-1)^{\sigma_b^{(v_1(e'))} \sigma_b^{(v_2(e'))}} \\ &= (-1)^{|E_c| + \sum_{\mu} |E_\mu|} \bigotimes_{\mu} \bigotimes_{e \in E_\mu} \sigma_{a_\mu}^{(u(e))} \sigma_{a_\mu}^{(v(e))} \\ &\quad \times \bigotimes_{e' \in E_c} \sigma_b^{(v_1(e'))} \sigma_b^{(v_2(e'))}. \end{aligned}$$

We take the following convention for b as reported in Table II. For POVM outcome $a_c = z$, we take $b = x$; for $a_c = x$, we take $b = z$; for $a_c = y$, we take $b = z$. With this choice we have

$$\begin{aligned} \mathcal{K}_C &= (-1)^{|E_c| + \sum_{\mu} |E_\mu|} \bigotimes_{\mu} (\bigotimes_{e \in E_\mu} \lambda_{u(e)}) Z_\mu^{|E_\mu|} \\ &\quad \times \bigotimes_{e \in E_\mu} \sigma_{a_\mu}^{v(e)} \sigma_b^{v(e)} X_c. \end{aligned}$$

$$\begin{aligned} \mathcal{K}_C &= (-1)^{|E_c| + \sum_{\mu} |E_\mu|} \bigotimes_{\mu} (\bigotimes_{e \in E_\mu} \lambda_{u(e)}) Z_\mu^{|E_\mu|} \\ &\quad \times \left(\bigotimes_{a_\mu \neq b} \bigotimes_{e \in E_\mu} \lambda_{v(e)} \right) Q_c, \\ Q_c &= \begin{cases} i^{n \neq b} X_c & \text{if } n \neq b \text{ is even} \\ -i^{1+n \neq b} (-1)^{\delta_{a_c, x}} Y_c & \text{if } n \neq b \text{ is odd} \end{cases} \end{aligned}$$

$$n \neq b \equiv \sum_{\mu, a_\mu \neq b} |E_\mu|$$

TABLE II. The choice of b and $a_{\mu \neq b}$.

a_c	z	x	y
b	x	z	z
$a_{\mu \neq b}$	y	y	x

POVM outcome	z	x	y
Stabilizer generator	$\lambda_i \lambda_j \sigma_z^{[i]} \sigma_z^{[j]}$	$\lambda_i \lambda_j \sigma_x^{[i]} \sigma_x^{[j]}$	$\lambda_i \lambda_j \sigma_y^{[i]} \sigma_y^{[j]}$
Logical \bar{X} operator	$\bigotimes_{j=1}^{4 C } \sigma_x^{[j]}$	$\bigotimes_{j=1}^{4 C } \sigma_z^{[j]}$	$\bigotimes_{j=1}^{4 C } \sigma_z^{[j]}$
Logical \bar{Z} operator	$\lambda_i \sigma_z^{[i]}$	$\lambda_i \sigma_x^{[i]}$	$\lambda_i \sigma_y^{[i]}$

Probability of POVM outcomes

- Measurement gives random outcomes, but what is the probability of a given set of outcomes?

$$P(\{\alpha(v)\}) \sim \langle \psi_{\text{AKLT}} | \bigotimes_v F_{\alpha(v)}^\dagger F_{\alpha(v)} | \psi_{\text{AKLT}} \rangle$$

- Can evaluate this using coherent states; alternatively use tensor product states
- Turns out to be a geometric object

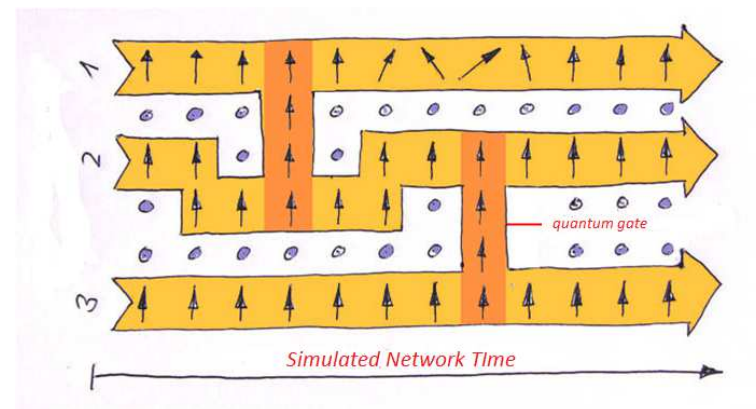
[**Wei**, Affleck & Raussendorf, PRL '11 & PRA '12]

$$P(\{\alpha(v)\}) \sim 2^{|V| - |\mathcal{E}|}$$

Difference from 1D case: graph & percolation

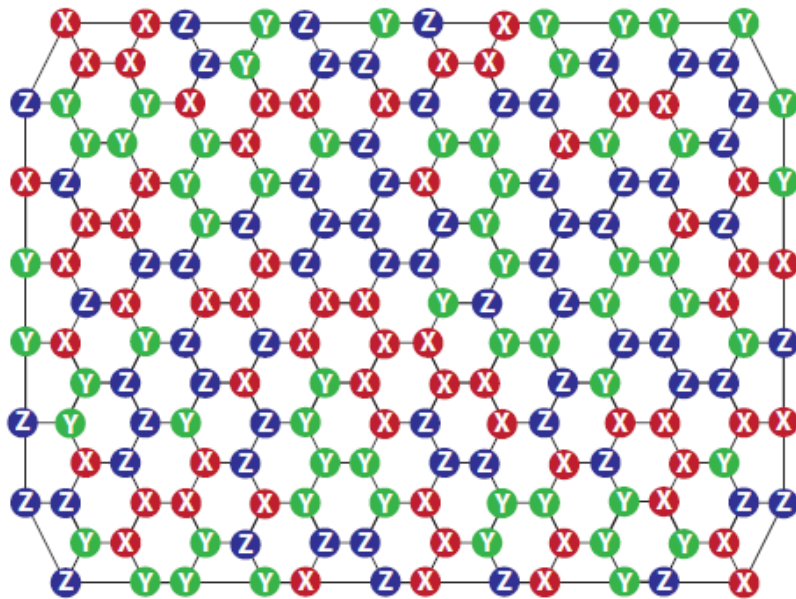
[*Wei, Affleck & Raussendorf PRL'11*]

1. What is the graph? which determines the graph state
→ How to identify the graphs ?
2. Are they percolated? (if so, universal resource)

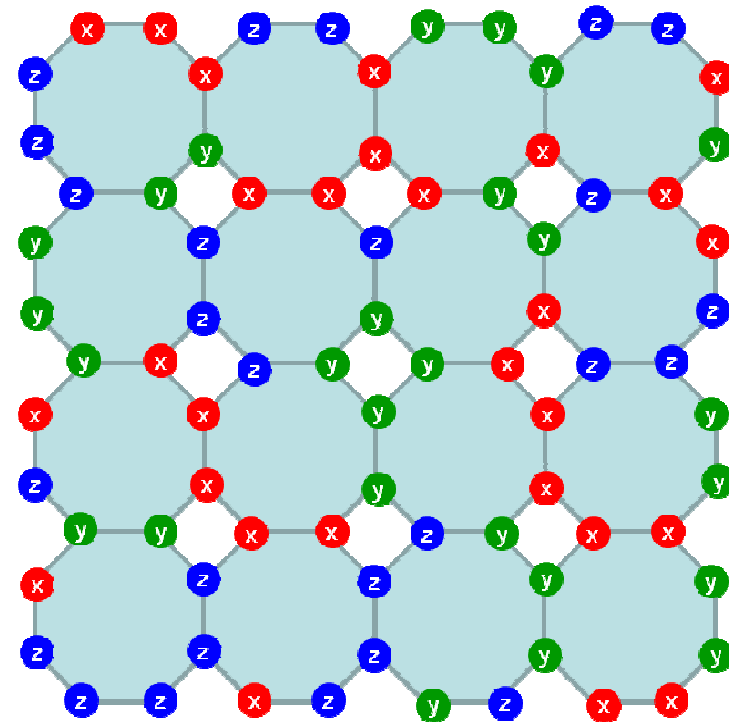


Recipe: construct graph for 'the graph state'

- Examples: random POVM outcomes x , y , z



honeycomb

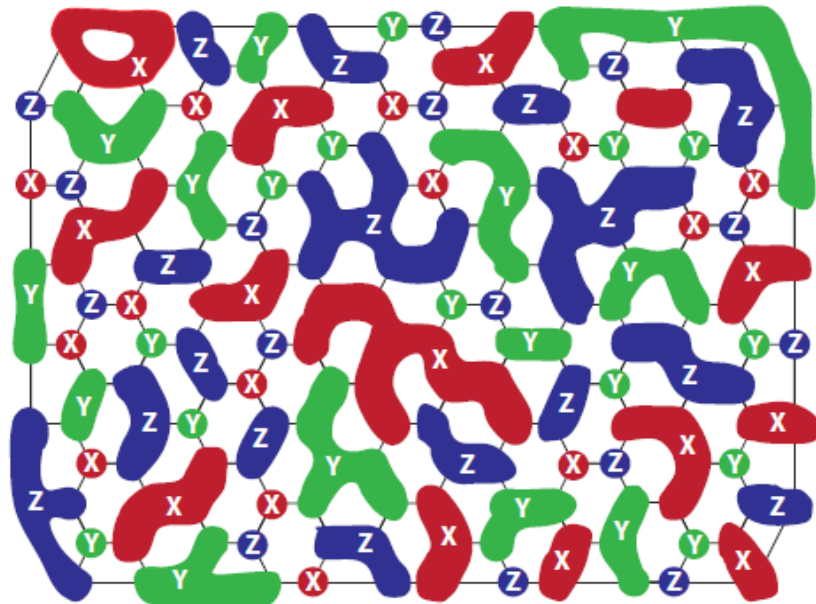


square octagon

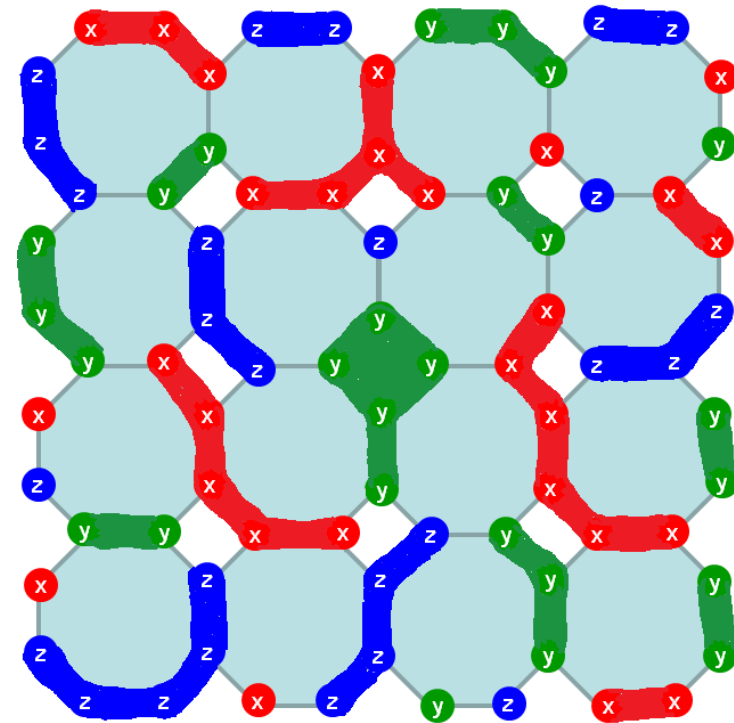
$$P(\{\alpha(v)\}) \sim 2^{|V|-|\mathcal{E}|}$$

Step 1: Merge sites to “domains” → vertices

➤ 1 domain = 1 logical qubit



honeycomb

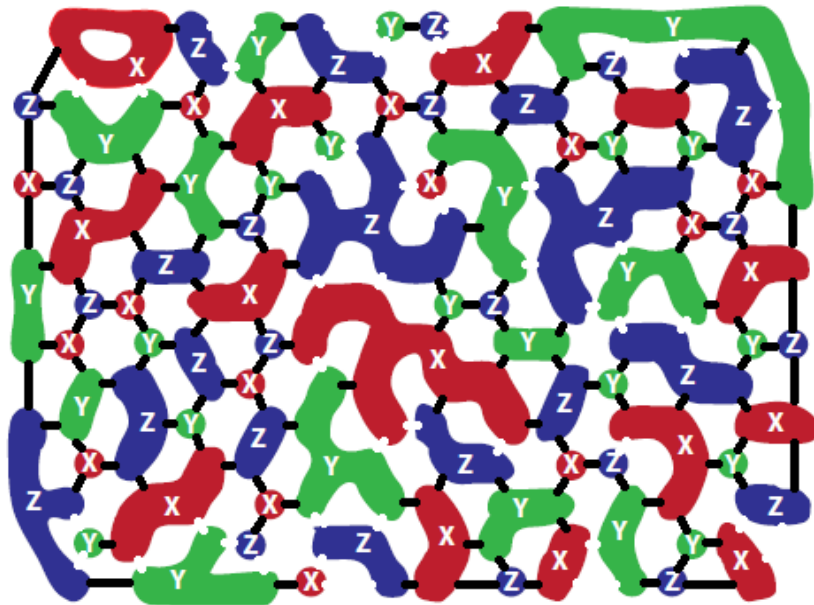


square octagon

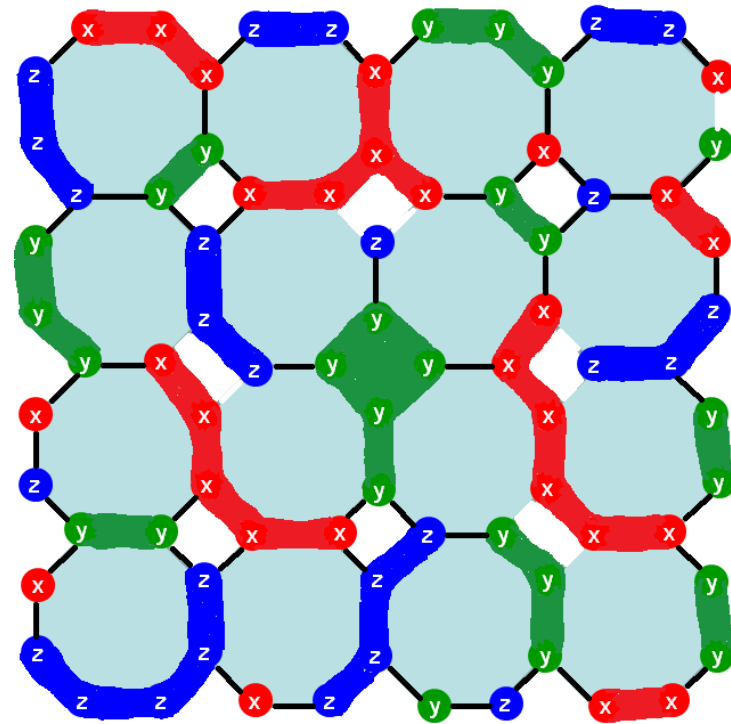
$|\uparrow\uparrow\uparrow\uparrow\rangle$ $|\downarrow\uparrow\uparrow\downarrow\rangle$: encoding of a logical qubit

Step 2: edge correction between domains

- Even # edges = 0 edge, Odd # edges = 1 edge
(due to $\sigma_z^2 = I$ in the C-Z gate)



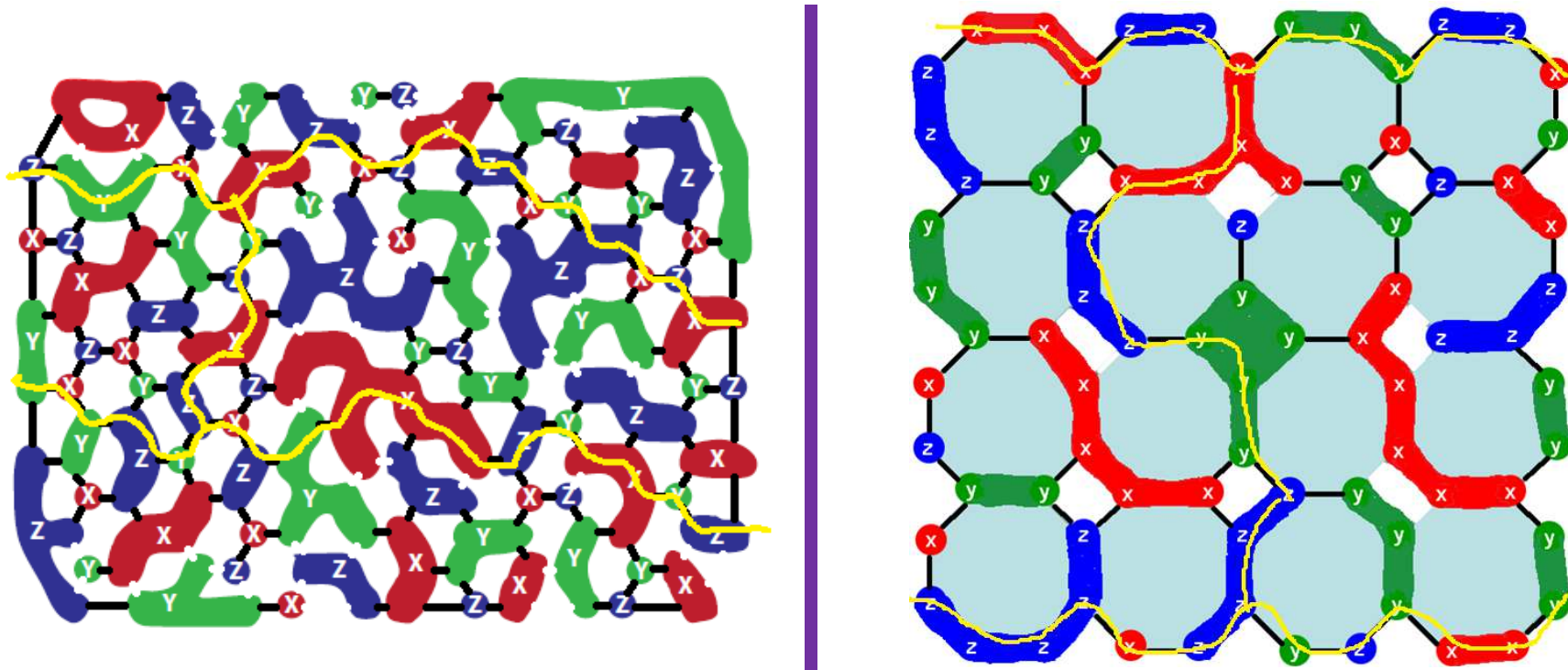
honeycomb



square octagon

Step 3: Check connections (percolation)

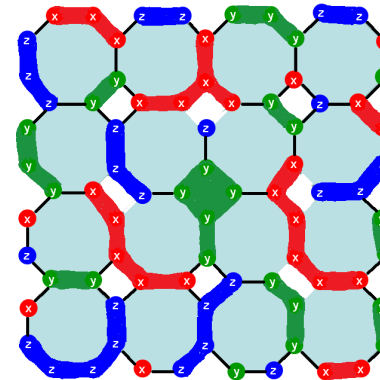
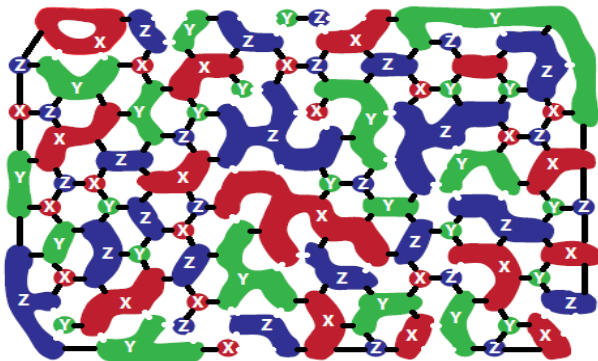
- Sufficient number of wires if graph is in supercritical phase (percolation)



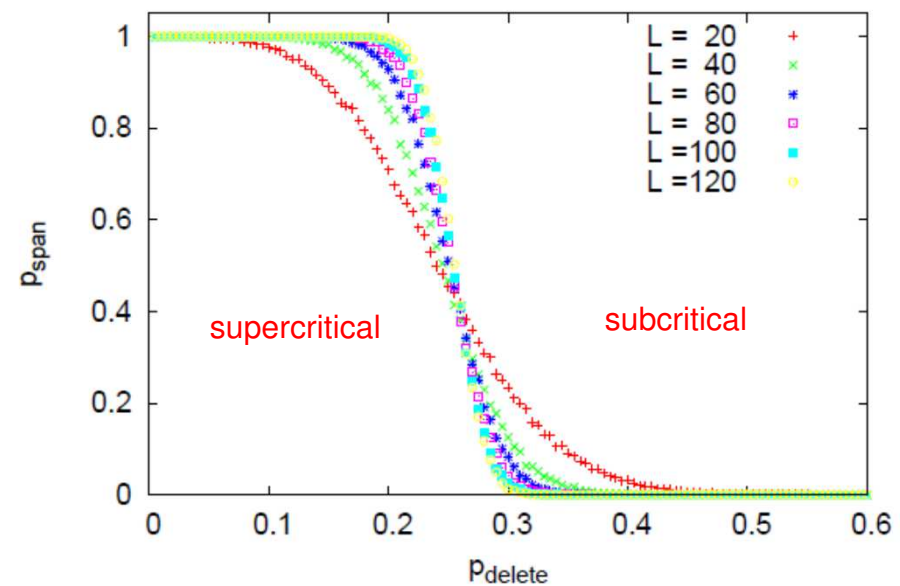
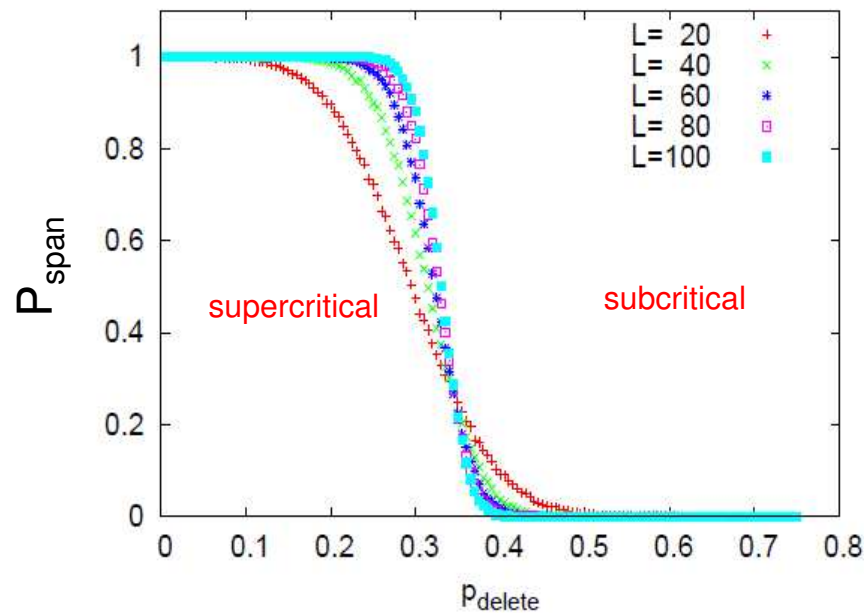
- ✓ Verified this for honeycomb, square octagon and cross lattices
➔ AKLT states on these are universal resources

How robust is connectivity?

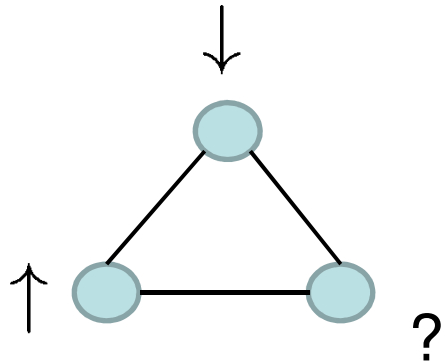
- Characterized by artificially removing domains to see when connectivity collapses (phase transition)



[Wei'13]



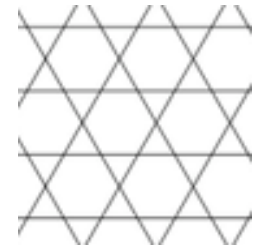
Frustration on star lattice



→ Cannot have POVM outcome
 xxx , yyy or zzz on a triangle

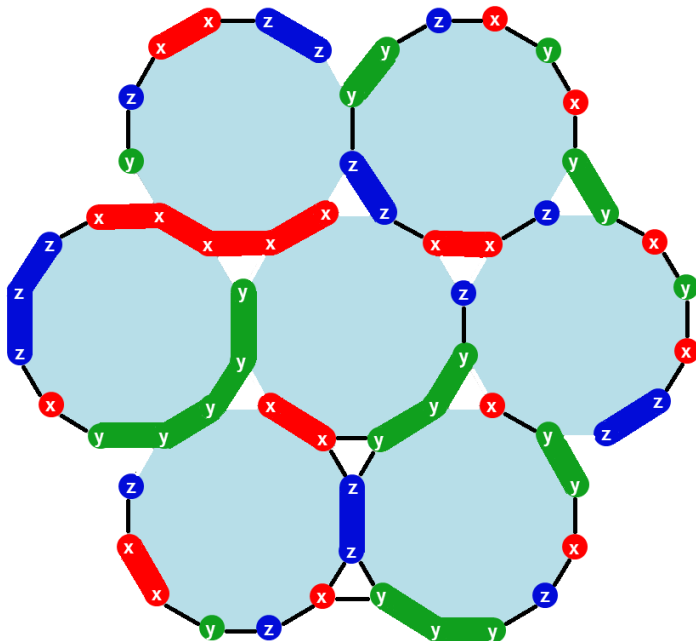
→ Consequences:

- (1) Only 50% edges on triangles occupied
 $< p_{th} \approx 0.5244$ of Kagome
→ disconnected graph



- (2) Simulations confirmed: graphs not percolated

→ AKLT on star likely NOT universal



Difficulty for spin-2

- Technical problem: trivial extension of POVM does NOT work!

$$\begin{aligned} F_z &= |2\rangle\langle 2|_z + |-2\rangle\langle -2|_z \\ F_x &= |2\rangle\langle 2|_x + |-2\rangle\langle -2|_x \\ F_y &= |2\rangle\langle 2|_y + |-2\rangle\langle -2|_y \end{aligned}$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I$$

→ Leakage out of logical subspace (error)

- Fortunately, can add elements K's to complete the identity

$$\left\{ \begin{aligned} F_\alpha &= \sqrt{\frac{2}{3}} (|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|) & [\text{Wei, Haghnegahdar, Raussendorf'14}] \\ K_\alpha &= \sqrt{\frac{1}{3}} (|\phi_\alpha^-\rangle\langle\phi_\alpha^-|) & |\phi_\alpha^-\rangle \equiv \sqrt{\frac{1}{2}} (|S_\alpha = 2\rangle - |S_\alpha = -2\rangle) \\ \alpha &= x, y, z \end{aligned} \right.$$

Completeness: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

Another difficulty: sample POVM outcomes

$$p(\{F, K\}) = \langle \text{AKLT} | \bigotimes_u F_{\alpha(u)}^\dagger F_{\alpha(u)} \bigotimes_v K_{\beta(v)}^\dagger K_{\beta(v)} | \text{AKLT} \rangle = ? \quad [\text{Wei, Raussendorf '15}]$$

□ How to calculate such an N -body correlation function?

Lemma. If there exists a set Q (subset of D_K) such that $-\bigotimes_{\mu \in Q} (-1)^{|V_\mu|} X_\mu$ is in the stabilizer group $\mathcal{S}(|G_0\rangle)$ of the state $|G_0\rangle$, then $p(\{F, K\}) = 0$. Otherwise,

$$p(\{F, K\}) = c \left(\frac{1}{2} \right)^{|\mathcal{E}| - |V| + 2|J_K| - \dim(\ker(H))},$$

where c is a constant.

$$\left\{ \begin{array}{l} |G_0\rangle \sim \bigotimes_v F_{\alpha(v)} | \text{AKLT} \rangle \\ D_K: \text{ set of domains having all sites POVM } K \\ (H)_{\mu\nu} = 1 \text{ if } \{K_\mu, X_\nu\} = 0, \text{ and } (H)_{\mu\nu} = 0 \text{ otherwise} \end{array} \right.$$

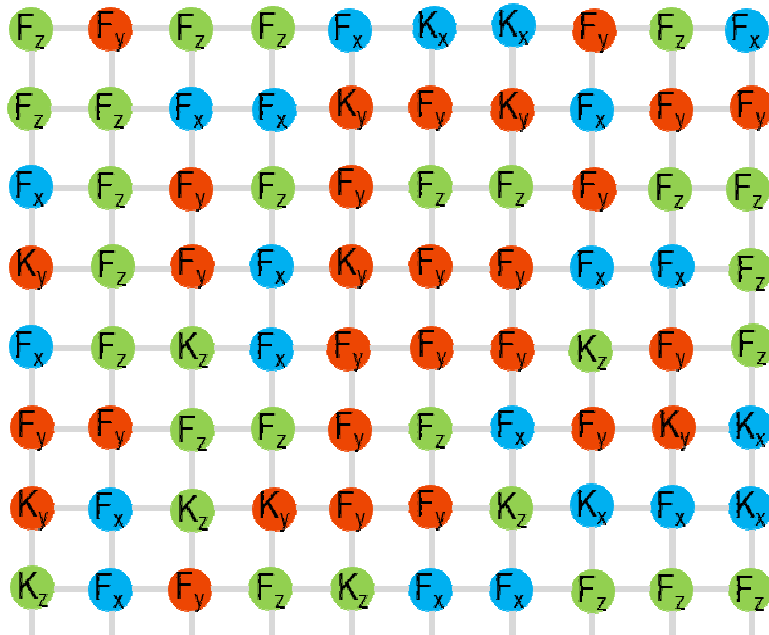
→ Bottom line: can use Monte Carlo sampling

Local POVM: 5-level to (2 or 1)-level

$$\left\{ \begin{array}{l} F_\alpha = \sqrt{\frac{2}{3}} (|S_\alpha = +2\rangle \langle S_\alpha = +2| + |S_\alpha = -2\rangle \langle S_\alpha = -2|) \quad [\text{Wei, Haghnegahdar, Raussendorf '14}] \\ K_\alpha = \sqrt{\frac{1}{3}} (|\phi_\alpha^-\rangle \langle \phi_\alpha^-|) = \frac{1}{\sqrt{2}} |\phi_\alpha^-\rangle \langle \phi_\alpha^-| F_\alpha \quad |\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}} (|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle) \\ \alpha = x, y, z \end{array} \right.$$

Completeness: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

□ POVM gives random outcome $F_x, F_y, F_z, K_x, K_y, K_z$ at each site



→ Local action (depends on outcome):

$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z} |\Phi\rangle$$

or

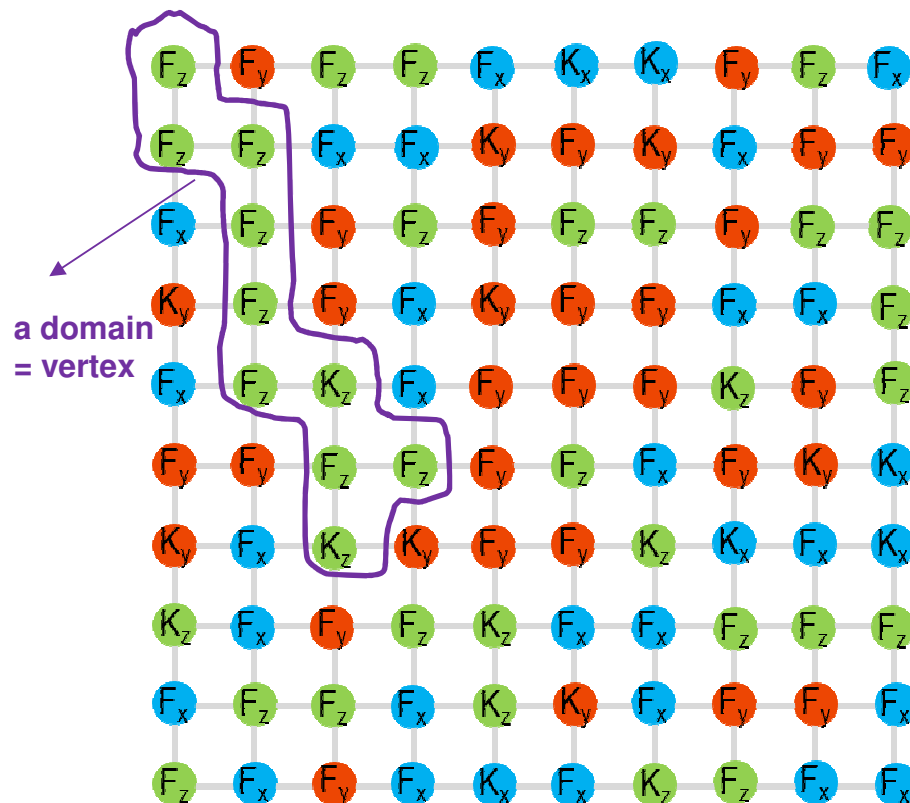
$$|\Phi\rangle \longrightarrow K_{\alpha=x,y,\text{or } z} |\Phi\rangle$$

Post-POVM state: graph state

$$\left\{ \begin{array}{l} F_\alpha = \sqrt{\frac{2}{3}} (|S_\alpha = +2\rangle \langle S_\alpha = +2| + |S_\alpha = -2\rangle \langle S_\alpha = -2|) \\ K_\alpha = \sqrt{\frac{1}{3}} (|\phi_\alpha^-\rangle \langle \phi_\alpha^-|) = \frac{1}{\sqrt{2}} |\phi_\alpha^-\rangle \langle \phi_\alpha^-| F_\alpha \end{array} \right. \quad [Wei, Haghnegahdar, Raussendorf'14]$$

$$|\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}} (|S_\alpha = 2\rangle \pm |S_\alpha = -2\rangle)$$

$\alpha = x, y, z$



- If F outcome on **all** sites
→ a *planar* graph state

$$|G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)} |\text{AKLT}\rangle$$

- ✓ **Vertex** = a domain of sites with same color (x, y or z)

- K outcome = F followed by ϕ^\pm measurement (then *post-selecting* '-' result)

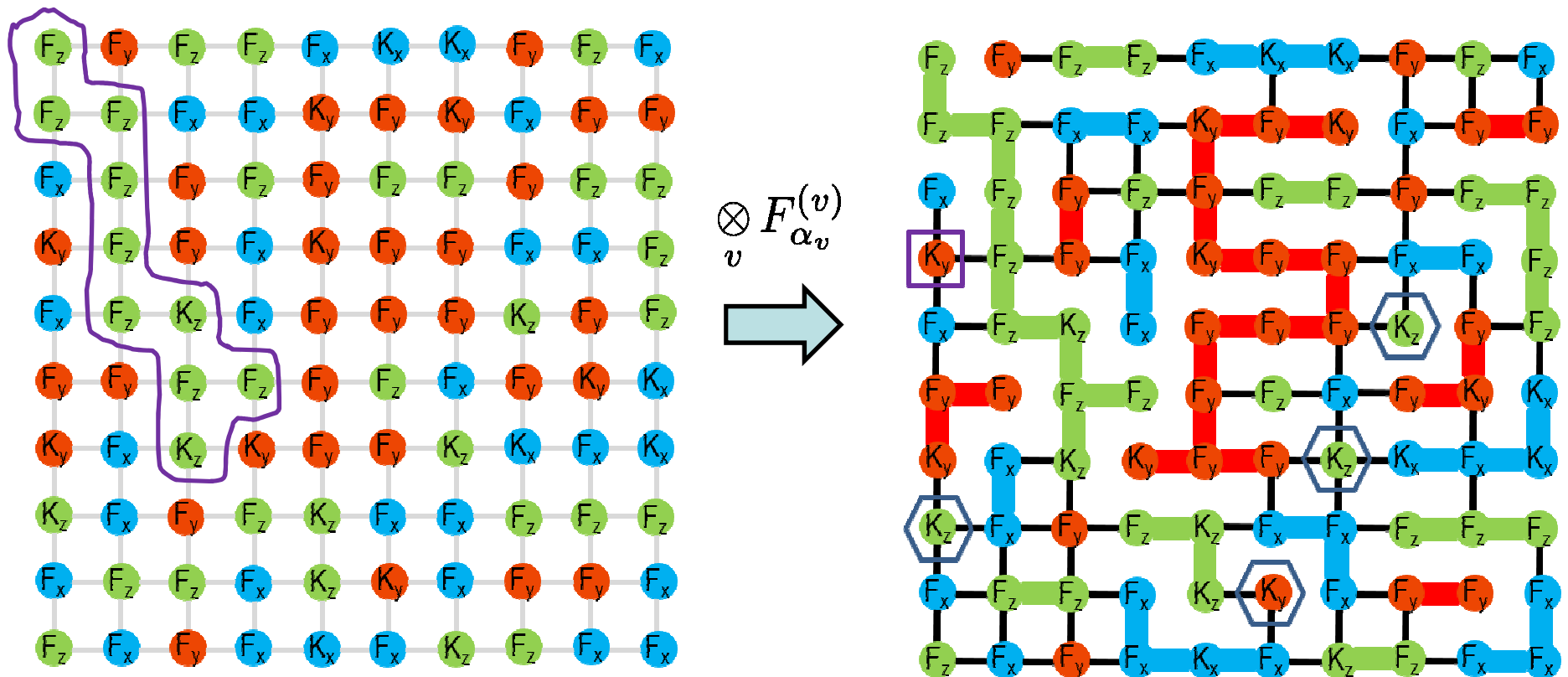
→ **Either**

- (1) shrinks domain size [trivial] or
- (2) logical X or Y measurement [**nontrivial**]

POVM \rightarrow Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)

$$|G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)} |\text{AKLT}\rangle$$



□ Effect of nontrivial $K_\alpha = \frac{1}{\sqrt{2}} |\phi_\alpha^-\rangle \langle \phi_\alpha^-| F_\alpha$
 \rightarrow non-planar graph

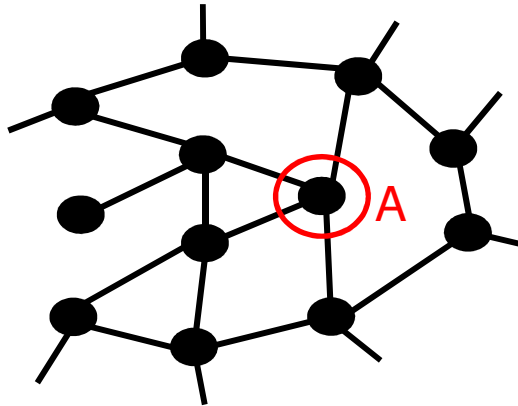
□ :logical X
measurement

⬡ :logical Y
measurement

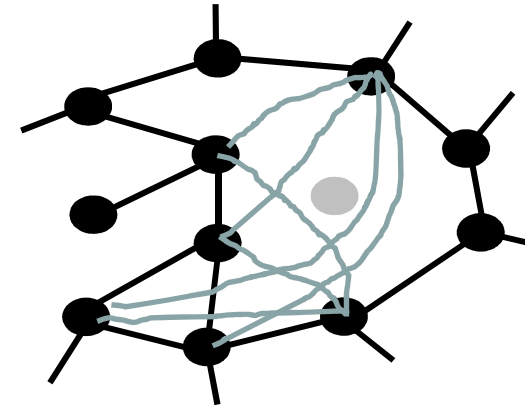
Non-planarity from X/Y measurement

[See e.g. *Hein et '06*]

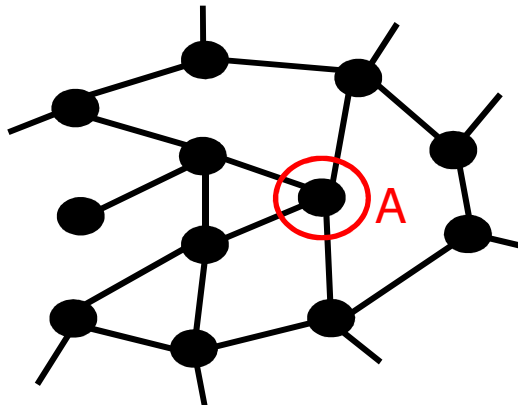
X:



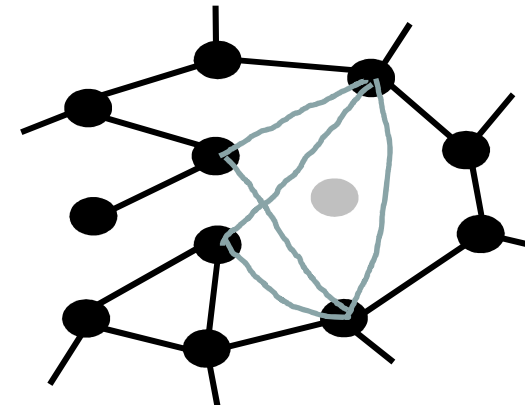
X measurement
on A



Y:



Y measurement
on A



➔ Effect of X measurement is more complicated than Y measurement

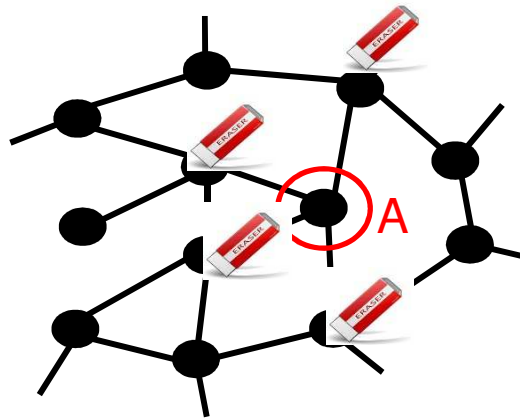
Restore planarity: further measurement

- Deal with non-planarity due to Pauli **X** measurement:

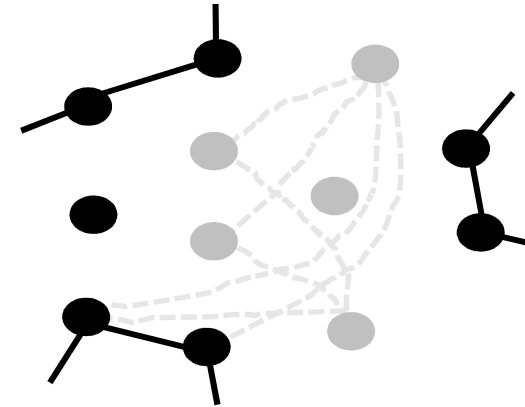
remove all vertices surrounding that of X measurement (via Z measurement)



X:



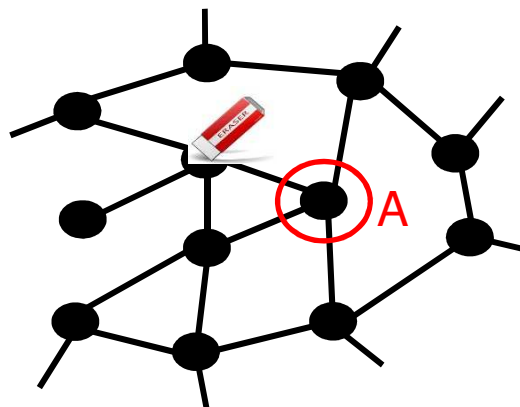
X measurement
on A



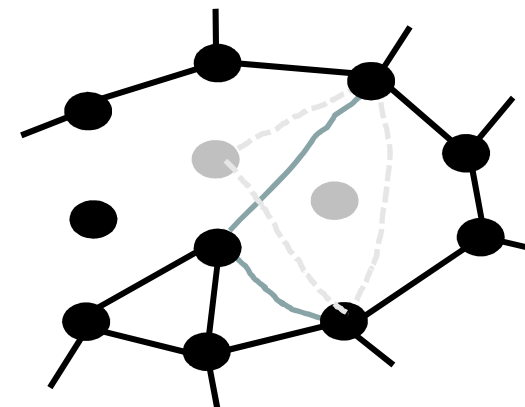
- Deal with non-planarity due to Pauli **Y** measurement:

remove only subset of vertices surrounding that of Y measurement

Y:

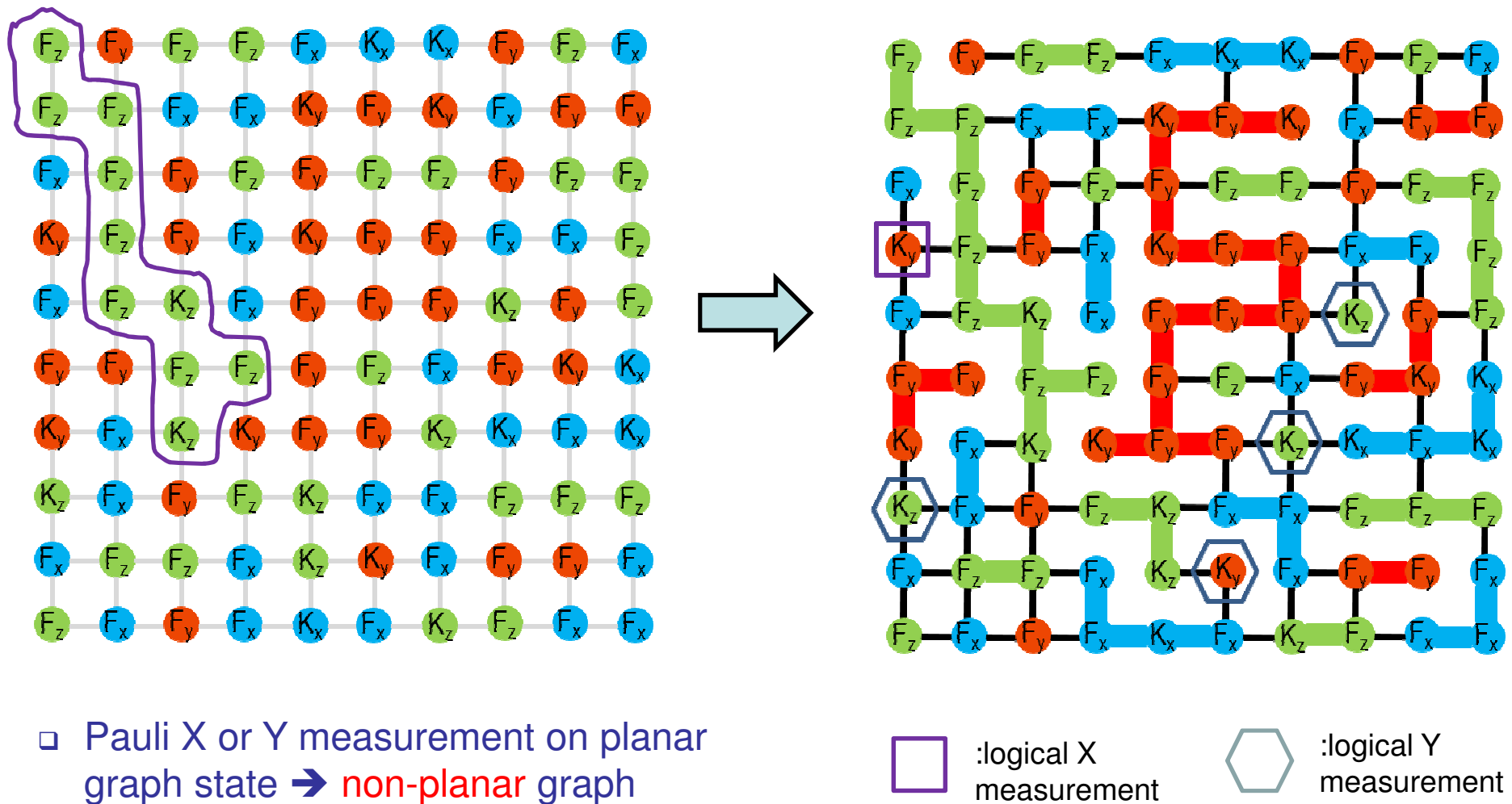


Y measurement
on A

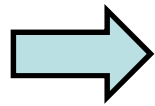


POVM \rightarrow Graph of the graph state

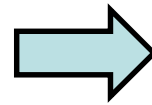
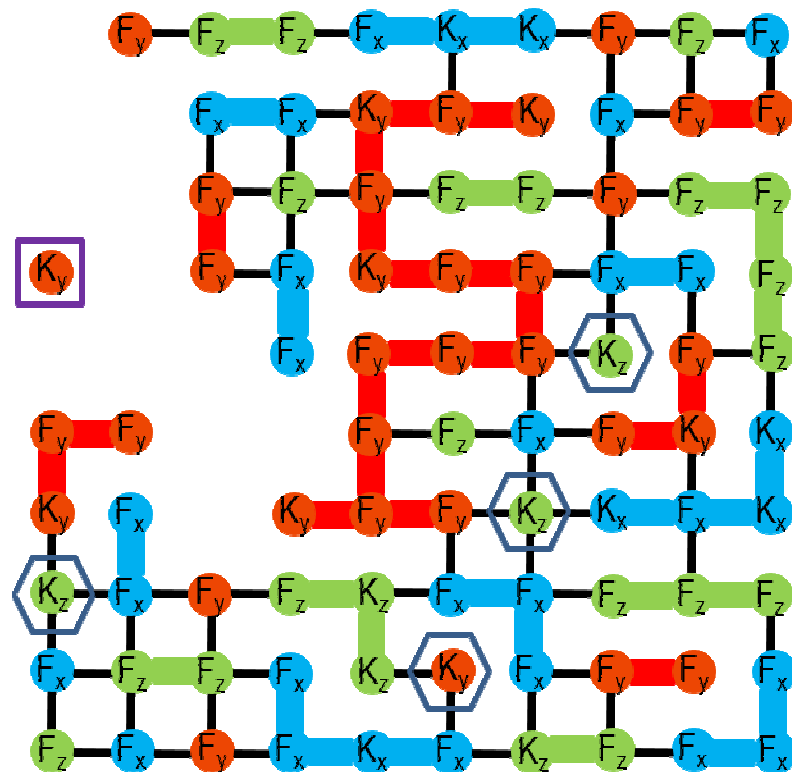
Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)



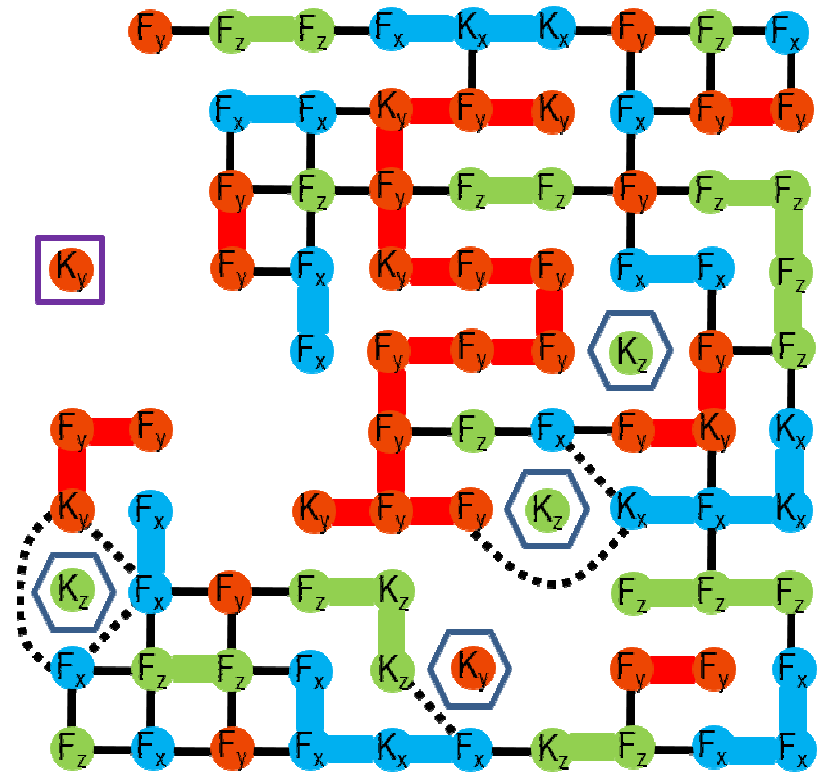
Restore Planarity by Another round of measurement



Deal with X measurement

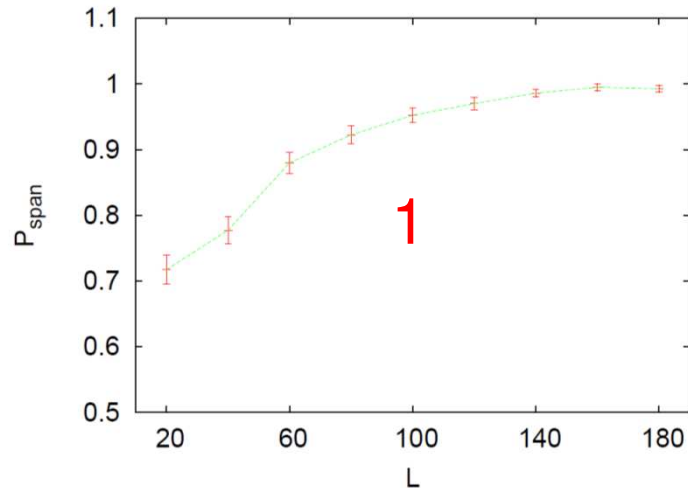


Deal with Y measurement

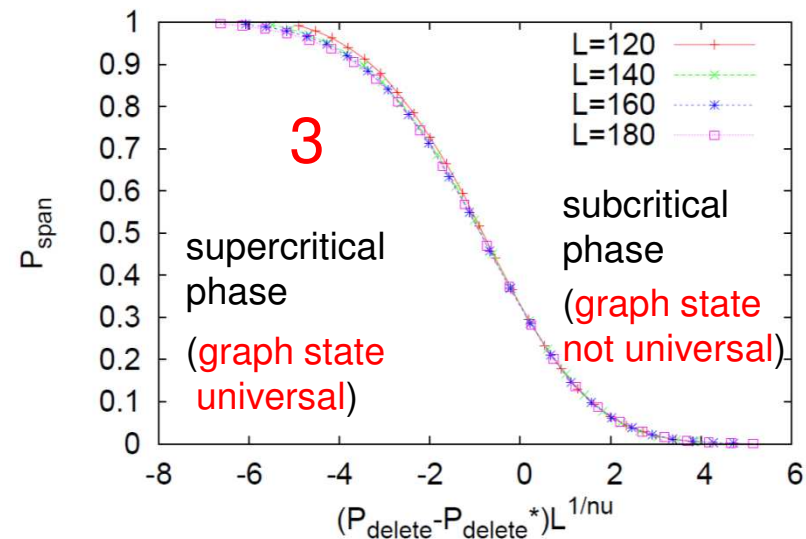
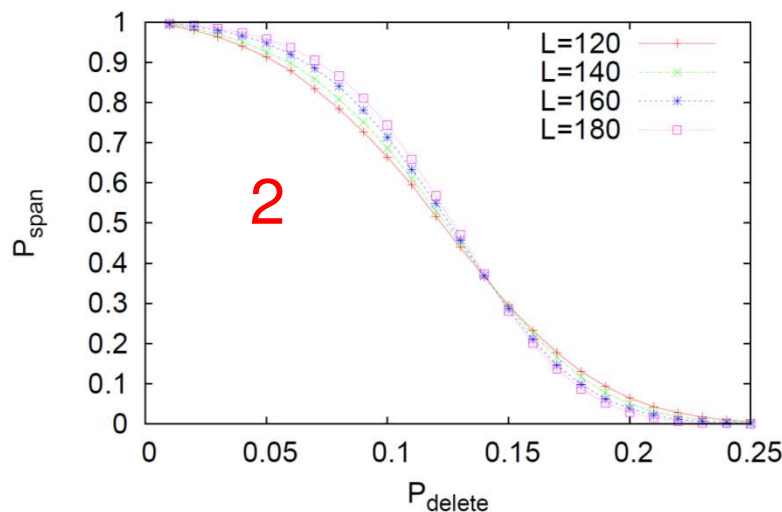


Examining percolation of typical graphs

(resulting from POVM and active logical Z measurement)



- ✓ 1. As system size $N=L \times L$ increases, exists a spanning cluster with high probability
- ✓ 2. Robustness of connectivity: finite percolation threshold (deleting each vertex with increasing probability)
- ✓ 3. Data collapse: verify that transition is continuous (critical exponent $\nu = 4/3$)

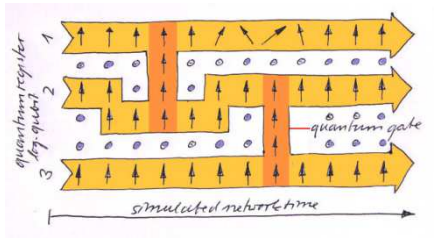


Spin-2 AKLT on square is universal for quantum computation

- Because the typical graph states (obtained from local measurement on AKLT) are universal → hence AKLT itself is universal
- Difference from spin-3/2 on honeycomb: ***not all*** randomly assigned POVM outcomes are allowed
→ weight formula is crucial
- Emerging (partial) picture for AKLT family:
AKLT states involving spin-2 and other lower spin entities are universal if they reside on a 2D frustration-free regular lattice with any combination of spin-2, spin-3/2, spin-1 and spin-1/2

Summary

□ Introduced one-way (cluster-state) quantum computation



→ Measurement-based QC uses entanglement

→ Teleportation viewpoint and tensor-network approach (correlation space QC)

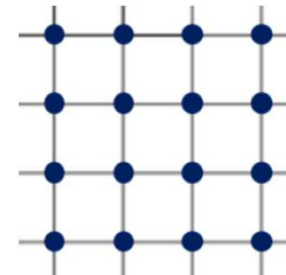
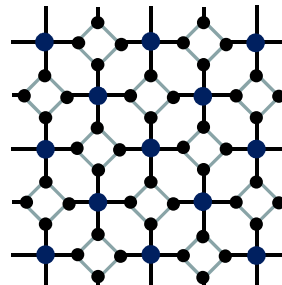
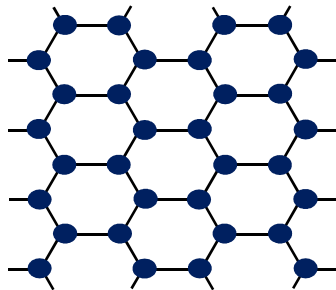
→ Universality in graph states

→ Fault tolerance & surface code

→ Blind quantum computation

→ Possible connection to SPT order

□ Showed various AKLT states (on different 2D lattices) provide universal resource for quantum computation



Not covered

- ❑ MBQC, classical spin models & complexity

[Van den Nest, Dur & Briegel '07, '08]

- ❑ Thermal phase diagram of MBQC

[Fujii, Nakata, Ohzeki & Murao '13]

[Li et al. '11, Wei, Li & Kwek '14]

- ❑ Deformed AKLT models & transition in QC power

*[Darmawan, Brennen & Bartlett '12,
Huang & Wei '16]*

- ❑ Verifiable blind QC

[Hayashi & Morimae '15]

Open problems

- ❑ Complete characterization of all universal resource states?
 - Even for AKLT family?
- ❑ Universal resource in an entire SPT phase?
 - Even for just 1D SPT phase and arbitrary 1-qubit gate?
- ❑ Deeper connection of topological QC to MBQC?

