When Is a Measurement Reversible?

Information Gain and Disturbance in Quantum Measurements (revisited)

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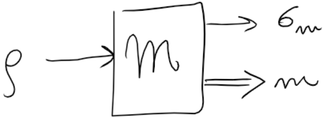
these slides are available for download at `tinyurl.com/BDW-aqis16`

Quantum Measurement Processes



✔ an operation transforming a quantum input (the density operator representing the state of the system before the measurement, i.e., the "pre-measurement state") into a classical output (the "outcome" of the measurement) and a quantum output (the state of the system conditional on the observation of some outcome, i.e., the "post-measurement state")

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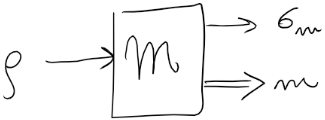

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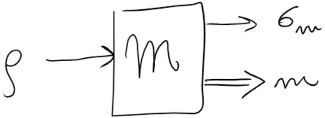

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- ✔ upon input ρ, the outcome m is obtained with probability $p(m) = \mathrm{Tr}[\mathcal{E}_m(\rho)]$ and the corresponding post-measurement state is $\sigma_m = \mathcal{E}_m(\rho)/p(m)$

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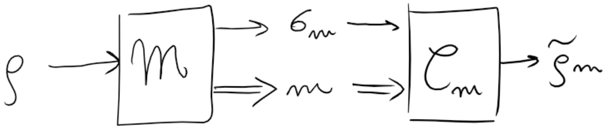

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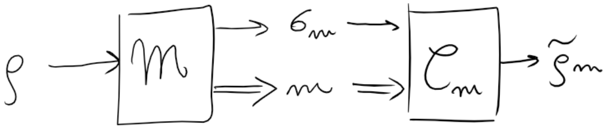

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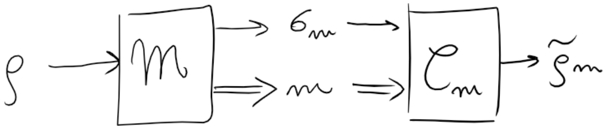

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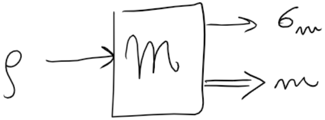
- **Fact.** Among all measurements with the same outcome statistics, efficient measurements (i.e., $\mathcal{E}_m(\cdot) = E_m \cdot E_m^\dagger$) are the least disturbing one: we will hence focus on these only



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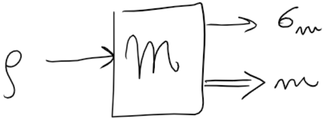

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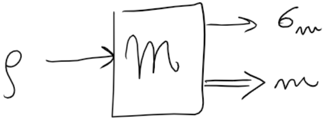

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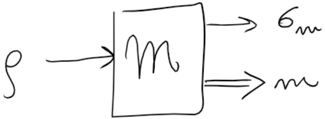

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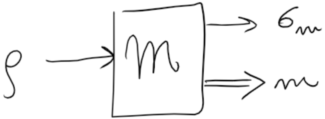

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- tempting analogy with work and heat in thermodynamics

The Technical Lemma

Theorem

Let Φ *be a subunital (i.e.,* $\Phi(I) \leqslant I$*) CPTP map. Then there exists a CPTP map* Ψ *such that, for any* ρ*,* $H[\Phi(\rho)] - H(\rho) \geqslant D[\rho \| (\Psi \circ \Phi)(\rho)]$*.*

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Stronger than any other (currently known) recoverability results:

1. the lemma holds also for positive (not necessarily completely positive) Φ
2. the explicit form of correction Ψ is known
3. $D(\rho \| \sigma) \geqslant D_{\mathbb{M}}(\rho \| \sigma) \geqslant -\log F(\rho, \sigma)$, where $F(\rho, \sigma) \triangleq \big\| \sqrt{\rho}\sqrt{\sigma} \big\|_1^2$

Application to Efficient Quantum Measurements

- ✔ given an efficient measurement process $\mathfrak{M} = \{E_m\}$, construct the CPTP map $\mathcal{M}(\cdot) = \sum_m E_m \cdot E_m^\dagger \otimes |m\rangle\langle m|$

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- in other words, the channel $\mathcal{M}$ is subunital and the Technical Lemma can be applied

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Theorem (Strengthened Second Law for Efficient Quantum Measurements)

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- ✔ hence, the "extrinsic noise" measures the irreversibility of efficient measurements (as does entropy for adiabatic processes)
- ✔ indeed, in [K. Jacobs, PRA **80**, 012322 (2009)] the inequality $H(M) - I_G(\rho, \mathfrak{M}) \geqslant 0$ is interpreted as the second law for quantum measurements; the above Theorem considerably strengthens this

Thank You

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