An infinite dimensional Birkhoff's Theorem and LOCC-convertibility

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Birkhoff's Theorem (: matrix analysis(math))
& in infinite dimensinaol Hilbert space
LOCC-convertibility (: quantum information)

Notation

- $\mathcal{H}$, $\mathcal{K}$: separable Hilbert spaces. (Unless specified otherwise $\dim = \infty$)
- $|\psi\rangle, |\phi\rangle \in \mathcal{H} \otimes \mathcal{K}$: unit vectors.
- majorization: for $\sigma = \sum_{n=1}^{\infty} a_n |x_n\rangle\langle x_n|$, $\rho = \sum_{n=1}^{\infty} b_n |y_n\rangle\langle y_n| \in \mathcal{S}(\mathcal{H})$,
  $\underline{\sigma \prec \rho} \underset{def}{\Longleftrightarrow} \sum_{i=1}^{n} a_i^{\downarrow} \leq \sum_{i=1}^{n} b_i^{\downarrow}, \forall n \in \mathbb{N}$.
- $|\psi\rangle \underset{LOCC}{\to} |\phi\rangle \underset{def}{\Longleftrightarrow} \exists\ n \in \mathbb{N} \cup \{\infty\}$, $\exists$ POVM on $\mathcal{H}$ $\{M_i\}_{i=1}^{n}$ and $\exists$ a set of unitary on $\mathcal{K}$ $\{U_i\}_{i=1}^{n}$ s.t.
  $$|\phi\rangle\langle\phi| = \sum_{i=1}^{n} (M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*), \text{ in } \mathfrak{C}_1(H).$$
- "in $\mathfrak{C}_1(H)$" means the convergence in Banach space $(\mathfrak{C}_1(\mathcal{H}), ||\cdot||_1)$ when $n = \infty$.

LOCC-convertibility[2/13]

Theorem(Nielsen, 1999)[1][2, S12.5.1] : the case $\dim \mathcal{H}, \dim \mathcal{K} < \infty$

$$|\psi\rangle \underset{LOCC}{\rightarrow} |\phi\rangle \iff \mathrm{Tr}_K |\psi\rangle\langle\psi| \prec \mathrm{Tr}_K |\phi\rangle\langle\phi|$$

Theorem(Owari et al, 2008)[3] : the case of $\dim \mathcal{H}, \dim \mathcal{K} = \infty$

$$|\psi\rangle \underset{LOCC}{\rightarrow} |\phi\rangle \Longrightarrow \mathrm{Tr}_K |\psi\rangle\langle\psi| \prec \mathrm{Tr}_K |\phi\rangle\langle\phi|$$

$$\mathrm{Tr}_K |\psi\rangle\langle\psi| \prec \mathrm{Tr}_K |\phi\rangle\langle\phi| \Longrightarrow |\psi\rangle \underset{\epsilon-LOCC}{\rightarrow} |\phi\rangle$$

- where " $\underset{\epsilon-LOCC}{\rightarrow}$ " means "with (for any small) ϵ error by LOCC".
- $\mathrm{Tr}_K |\psi\rangle\langle\psi| \prec \mathrm{Tr}_K |\phi\rangle\langle\phi| \Rightarrow |\psi\rangle \underset{LOCC}{\rightarrow} |\phi\rangle$ in infinite dimensional space has been open.
- Our main purpose is to give an answer to this open problem.

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[2]M A Nielesen, I L Chuang, *Quantum Computation and Quantum Information*, Cambrige University Press, Cambrige, 2000.
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Birkhoff theorem[3/13]

- $\mathrm{Tr}_K|\psi\rangle\langle\psi| \prec \mathrm{Tr}_K|\phi\rangle\langle\phi| \Rightarrow |\psi\rangle \underset{LOCC}{\rightarrow} |\phi\rangle$ in finite dimensional space $(\star)$ is proved by Birkhoff's theorem.

Theorem(Birkhoff 1946)[4]

(i) $\mathrm{ex}\{d \times d$ doubly stochastic matrix$\} = \{d \times d$ permutation$\}$,

(ii) any doubly stochastic matrix can be represented as a finite convex combination of permutation matrices,

(iii) $\{d \times d$ doubly stochastic matrix$\} = \mathrm{co}\{d \times d$ permutation$\}$
$= \overline{\mathrm{co}}\{d \times d$ permutation$\}$.

- (ii) is used in the proof of $(\star)$.
- infinite dimensional analogue of Birkhoff's Theorem $\simeq$ Birkoff's problem 111
  - But, we can not use known results for Birkhoff's problem 111, since no one treated in any study (ii) in infinite dimensional space!

We construct an infinite dimensional analogue of (ii),
and using this, we want to get an infinite dimensional analogue of $(\star)$.

[4]G. Birkhoff. "Three observations on linear algebra".(Spanish) Univ. Nac. Tucuman. Revista A. 5 :137-151(1946)

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Definition of $\mathcal{D}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$ etc [4/13]

For a fixed CONS $(|i\rangle)_{i=1}^{\infty}$, we define the sets of bounded operators $\sum_{i,j=1}^{\infty} a_{ij}|i\rangle\langle j|$ as follows:

$$(\mathcal{P}(\mathcal{H})) : a_{ij} \in \{0,1\}, \sum_{j=1}^{\infty} a_{ij} = 1, \sum_{i=1}^{\infty} a_{ij} = 1 (\forall i,j) \leftarrow \text{permutation}$$

$$(\mathcal{P}_0(\mathcal{H})) : a_{ij} \in \{0,1\}, \sum_{j=1}^{\infty} a_{ij} \leq 1, \sum_{i=1}^{\infty} a_{ij} \leq 1 (\forall i,j) \leftarrow \text{subpermutation}$$

$$(\mathcal{D}(\mathcal{H})) : a_{ij} \in [0,1], \sum_{j=1}^{\infty} a_{ij} = 1 and \sum_{i=1}^{\infty} a_{ij} = 1 (\forall i,j) \leftarrow \text{doubly stochastic}$$

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remark

Any element of $\mathcal{P}(\mathcal{H})$ is unitary on $\mathcal{H}$.
This is a generalization of the fact that any permutation matrix is unitary.

Theorem1(Asakura)

For a separable infinite dimensional $\mathcal{H}$, we have

(i) $\mathrm{ex}\,\mathcal{D}(\mathcal{H}) = \mathcal{P}(\mathcal{H})$.

(ii) [integral representation of $\mathcal{D}(\mathcal{H})$] For any $D \in \mathcal{D}(\mathcal{H})$, there exist a probability measure μ_D on $\mathcal{P}(\mathcal{H})$ such that

$$D = WOT\text{-}\int_{\mathcal{P}(\mathcal{H})} X d\mu_D(X).$$

(iii) $\mathrm{co}\,\mathcal{P}(\mathcal{H}) \subsetneq \mathcal{D}(\mathcal{H}) \subsetneq \overline{\mathrm{co}}^w(\mathcal{P}(\mathcal{H})) = \mathcal{D}_0(\mathcal{H})$.

Theorem(Birkhoff)[rewrite] : the case $\mathcal{H} = \mathbb{C}^d$ and $(|i\rangle)_i$ =the standard basis in $\mathbb{C}^d$

(i) $\mathrm{ex}\,\mathcal{D}(\mathcal{H}) = \mathcal{P}(\mathcal{H})$.
(ii) For any $D \in \mathcal{D}(\mathcal{H})$, there exists a probability $\{p_i\}_{i=1}^{d!}$ s.t. $D = \sum_{i=1}^{d!} p_i P_i$, where $\{P_i\}_{i=1}^{d!} := \mathcal{P}(\mathcal{H})$
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Main results 2: a sufficient condition in infinite dim. [6/13]

Using Theorem 1 (ii), we get the following:

Theorem2(Asakura)

Let $|\psi\rangle$ and $|\phi\rangle$ be full rank unit vectors. If $\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}}|\phi\rangle\langle\phi|$, then

- there exist
  - a probability measure μ_D on $\mathcal{P}(\mathcal{H})$,
  - a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$
  - a set of densely defined unbounded operators $\{M_X\}_{X\in\mathcal{P}(\mathcal{H})}$ on $\mathcal{H}$ with $D(M_X) \supset \mathcal{H}_0 (\forall X \in \mathcal{P}(\mathcal{H}))$ satisfying "some conditions" such that

$$|\phi\rangle\langle\phi| = \int_{\mathcal{P}(\mathcal{H})} (M_X \otimes X^*)|\psi\rangle\langle\psi|(M_X^* \otimes X) d\mu_D(X), \text{in } \mathfrak{C}_1(\mathcal{H}) \quad (a)$$

  and $\int_{\mathcal{P}(\mathcal{H})} \langle\eta|M_X^* M_X|\xi\rangle d\mu_D(X) = \langle\eta|\xi\rangle \ \forall \eta, \xi \in \mathcal{H}_0$ (b) hold.

- In general case, we get the same result up to a local partial isometry.
- (a) is a generalization of $|\phi\rangle\langle\phi| = \sum_{i=1}^n (M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*)$.
- (b) is a generalization of $\sum_i M_i^* M_i = I_{\mathcal{H}}$.

sketch proof of the sufficient condition (Thm2) using the integral representation (Thm1(ii))[7/13]

We may assume that $|\psi\rangle$, $|\phi\rangle$ have a same Schmidt CONS.

$\rightarrow$ $|\psi\rangle = \sum_{i=1}^{\infty} \sqrt{a_i}|ii\rangle$, $|\phi\rangle = \sum_{i=1}^{\infty} \sqrt{b_i}|ii\rangle$ with $a := (a_i) \prec b := (b_i)$

$\rightarrow$ By [17], $\exists D \in \mathcal{D}(l^2)$ such that $a = Db$.

$\rightarrow$ By Theorem 1(ii), $\exists$ a prob. measure μ_D on $\mathcal{P}(\mathcal{H})$ s.t. $D = \int_{\mathcal{P}(l^2)} X d\mu_D(X)$.

Let $\rho_\psi := \mathrm{Tr}_K|\psi\rangle\langle\psi| = \sum_{i=1}^{\infty} a_i|i\rangle\langle i|$, $\rho_\phi := \mathrm{Tr}_K|\phi\rangle\langle\phi| = \sum_{i=1}^{\infty} b_i|i\rangle\langle i|$, then

$\rightarrow$ $\rho_\psi = \int_{\mathcal{P}(\mathcal{H})} X\rho_\phi X^* d\mu_D(X)$ in $\mathfrak{C}_1(H)$.

Then $M_X := \sqrt{\rho_\phi} X^*({\rho_\psi}^{-\frac{1}{2}})$ with $D(M_X) := D(\rho_\psi^{-\frac{1}{2}})$ and $\mathcal{H}_0 := \mathrm{span}\{|i\rangle\}_i$ satisfy the conditions. In particular,

- $(M_X \otimes X^*)|\psi\rangle = |\phi\rangle$, $\forall X \in \mathcal{P}(\mathcal{H})$

$\rightarrow$ $|\phi\rangle\langle\phi| = \int_{\mathcal{P}(\mathcal{H})}(M_X \otimes X^*)|\psi\rangle\langle\psi|(M_X^* \otimes X)d\mu_D(X), \mathrm{in}\, \mathfrak{C}_1(\mathcal{H})$

remark

The above M_X is understood in terms of relative modular operator.

[17]V. Kaftal, G. Weiss, "An infinite dimensional Schur-Horn Theorem and majorization theory", Journal of Functional Analysis, Vol. 259, No. 13, ;3115-3162(2010)

Main results 3: a characterization in infinite dim. [8/13]

Theorem 3(Asakura)

For full rank unit vectors $|\psi\rangle$ and $|\phi\rangle$, the following are equivalent:

(I) There exist

- a Borel set I of a certain of metric space,
- a probability measure μ on I,
- a set of densely defined (unbounded) operator $\{M_i\}_{i\in I}$ on $\mathcal{H}$ and a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ with $D(M_i) \supset \mathcal{H}_0$ and $(\sharp)$
- a set of unitary op $\{U_i\}_{i\in I}$ on $\mathcal{K}$

satisfying "some conditions" such that

$$|\phi\rangle\langle\phi| = \left(\int_I (M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*)\,d\mu(i)\right), \text{in } \mathfrak{C}_1(\mathcal{H}).$$

(II) $\mathrm{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}}|\phi\rangle\langle\phi|$ holds.

- (II)⇒(I) is a corollary of Theorem 2.
- (I)⇒(II) is proved by some arguments used in [21].

[21]Y. Li, P. Busch, "Von Neumann entropy and majorization", Journal of Mathematical Analysis and Applications, Vol. 408, :384-393(2012)

Theorem 3'(Asakura)

For unit vectors $|\psi\rangle$ and $|\phi\rangle$, the following are equivalent:

(I) There exist

- infinite rank partial isometry operators $V_{\mathcal{H}}$, $V_{\mathcal{K}}$,
- a Borel set I of a certain of metric space,
- a probability measure μ on I,
- a set of densely defined (unbounded) operator $\{M_i\}_{i\in I}$ on $\mathcal{H}$ and a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ with $D(M_i) \supset \mathcal{H}_0$ and $(\sharp)$
- a set of unitary op $\{U_i\}_{i\in I}$ on $\mathcal{K}$

satisfying "some conditions" such that

$$|\phi\rangle\langle\phi| = (V_{\mathcal{H}} \otimes V_{\mathcal{K}}) \left(\int_I (M_i \otimes U_i) |\psi\rangle\langle\psi| (M_i^* \otimes U_i^*) d\mu(i) \right) (V_{\mathcal{H}}^* \otimes V_{\mathcal{K}}^*), \text{ in } \mathfrak{C}_1(\mathcal{H}).$$

(II) $\mathrm{Tr}_{\mathcal{K}} |\psi\rangle\langle\psi| \prec \mathrm{Tr}_{\mathcal{K}} |\phi\rangle\langle\phi|$ holds.

- Theorem 3' immediately follows from Theorem 3.

Theorem1(ii)[rewrite]

For any $D \in \mathcal{D}(\mathcal{H})$, there exist a probability measure μ_D on $\mathcal{P}(\mathcal{H})$ such that

$$D = WOT\text{-}\int_{\mathcal{P}(\mathcal{H})} X d\mu_D(X).$$

Preliminary fact

- Let $\mathfrak{B}(\mathcal{H})_1 := \{X \in \mathfrak{B}(\mathcal{H}) | \|X\| \leq 1\}$, then $(\mathfrak{B}(\mathcal{H})_1, WOT)$ is a metrizable compact space [13, 4.6]
  - $\rightarrow$ Since $\mathcal{D}(\mathcal{H})$, $\mathcal{D}_0(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$, $\mathcal{P}_0(\mathcal{H})$ are all subsets of $\mathfrak{B}(\mathcal{H})_1$, we can consider these sets as (sub)sets in the compact metric space.

Convex theory

- Choquet theory : Any element of a metrizable compact convex subset $\mathcal{X}$ in a locally convex linear space has an integral representation on ex $\mathcal{X}$.
- A subset $\mathcal{Y}$ of a convex set $\mathcal{Z}$ is a face (of $\mathcal{Z}$),
  if $\lambda y_1 + (1-\lambda) y_2 \in \mathcal{Y}$ $(y_1, y_2 \in \mathcal{Z},\ \lambda \in [0,1]) \Rightarrow y_1, y_2 \in \mathcal{Y}$.
  (face $\simeq$ "set analogue of extreme point")

sketch proof of the integral representaion (cont.) [11/13]

The key of the proof is the following:

(I) $\overline{\mathcal{D}(\mathcal{H})}^{w} = \mathcal{D}_0(\mathcal{H})$ and $\mathcal{D}_0(\mathcal{H})$ is compact in WOT.

(II) $\mathcal{D}(\mathcal{H}) \subset \mathcal{D}_0(\mathcal{H})$ is face. $\mathrm{ex}\,\mathcal{D}(\mathcal{H}) = \mathcal{P}(\mathcal{H}) \subset \mathcal{P}_0(\mathcal{H}) = \mathrm{ex}\,\mathcal{D}_0(\mathcal{H})$.

Definition of $\mathcal{D}(\mathcal{H})$ and $\mathcal{D}_0(\mathcal{H})$[rewrite]

$\mathcal{D}(\mathcal{H})$: the set of doubly stochastic operator $\sum_{i,j} a_{ij}|i\rangle\langle j|$

$(\sum_i a_{ij} = \sum_j a_{ij} = 1)$

$\mathcal{D}_0(\mathcal{H})$: the set of doubly substochastic operator $\sum_{i,j} a_{ij}|i\rangle\langle j|$

$(\sum_i a_{ij}, \sum_j a_{ij} \leq 1)$

sketch of the proof

- By (I), we can apply Choquet's theorem to $\mathcal{D}_0(\mathcal{H})$ ($\supset \mathcal{D}(\mathcal{H})$) with WOT.
- Main subject is not $\mathcal{D}_0(\mathcal{H})$ but $\mathcal{D}(\mathcal{H})$.
- By (II), we can use the argement on $\mathcal{D}_0(\mathcal{H})$ for $\mathcal{D}(\mathcal{H})$.

sketch proof of the integral representaion (cont.)[12/13]

- By Choquet's theorem, for any $D \in \mathcal{D}(\mathcal{H}) \subset \mathcal{D}_0(\mathcal{H})$, there exists a probability measure μ_D on $\mathcal{P}_0(\mathcal{H})(= \mathrm{ex}\,\mathcal{D}_0(\mathcal{H}))$ such that
- $D = w\text{-}\int_{\mathcal{P}_0(\mathcal{H})} X d\mu_D(X)$
  $= w\text{-}\int_{\mathcal{P}(\mathcal{H})} X d\mu_D(X) + w\text{-}\int_{\mathcal{P}_0(\mathcal{H})\setminus\mathcal{P}(\mathcal{H})} X d\mu_D(X).$
- Thus, putting $p := \mu_D(\mathcal{P}(\mathcal{H}))$, $1 - p := \mu_D(\mathcal{P}_0(\mathcal{H}) \setminus \mathcal{P}(\mathcal{H}))$ and

$$\mathcal{D}(\mathcal{H}) \ni D = p \cdot \underbrace{w\text{-}\int_{\mathcal{P}(\mathcal{H})} p^{-1} X d\mu_D(X)}_{\star} + (1-p) \cdot \underbrace{w\text{-}\int_{\mathcal{P}_0(\mathcal{H})\setminus\mathcal{P}(\mathcal{H})} (1-p)^{-1} X d\mu_D(X)}_{\star\star} .$$

- $(\star) \in \mathcal{D}(\mathcal{H})$ and $(\star\star) \in \mathcal{D}_0(\mathcal{H}) \setminus \mathcal{D}(\mathcal{H})$.
- Since $\mathcal{D}(\mathcal{H})$ is a face of $\mathcal{D}_0(\mathcal{H})$ (:(b)), we get p=1 i.e.,

$$D = w\text{-}\int_{\mathcal{P}(\mathcal{H})} X d\mu_D(X). \quad \square$$

Conclusion [13/13]

Summary

- We establish the infinite dimensional Birkhoff's theorem with WOT
  - (The key is the situation that we can use some convex theories (face, Chouqet's thm...)).
- using this, we prove a new characterization of LOCC convertibility in infinite dimensional space
  - (The key is using a kind of relative modular operator).

Open Problem

- Can we get not integral form but discrete sum form?:

$$|\phi\rangle\langle\phi| = \sum_{i=1}^{\infty}(M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^* \otimes U_i^*)$$

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the case that D can not be written as discrete convex sum

The following $D \in \mathcal{D}(\mathcal{H})$ can not written as a discrete convex sum.

$$D := \oplus_{n=1}^{\infty} \frac{1}{n} 1_n,$$

where 1_n denotes the $n \times n$ matrix of ones, i.e., a $n \times n$ matrix where every element is equal to 1.

- Actually, this D can not be written a discrete convex sum, and then whenever we construct "POVM-element" M_X as described above, we can not make discrete LOCC form.
- But, if we can choose "good" M_X and U_X, for this D, we can construct discrete LOCC form (open problem).

example for not operator topology

In this slide, let $\mathcal{X}$ be the set of real infinite matrices where all sum of row and column absolutely converges.

- In [8], $\mathcal{X}$ is equipped with the weakest topology for which the following linear functional θ^is, ϕ^js and φ^{ij}s are all continuous.

$$\theta^i(X) := \sum_{j=1}^{\infty} x_{ij}, \quad \phi^j(X) := \sum_{i=1}^{\infty} x_{ij}, \quad \varphi^{ij}(X) := x_{ij}\,(i, j = 1, 2, \dots).$$

- In [9], $\mathcal{X}$ is equipped with the topology for which the following $V_{N,\epsilon}$s make a neighborhood basis of O,

$$V_{N,\epsilon} := \left\{X = (x_{ij}) \in \mathcal{X} \;\middle|\; \sum_{j=1}^{\infty} x_{ij} < \epsilon\,(i \leq N), \quad \sum_{i=1}^{\infty} x_{ij} < \epsilon\,(j \leq N)\right\}.$$

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relation to the result of Owari et al

Theorem(Owari et al, 2008)[3][rewrite]

If $\mathrm{Tr}_K|\psi\rangle\langle\psi| \prec \mathrm{Tr}_K|\phi\rangle\langle\phi|$, then there exist a LOCC sequence $\{\Lambda_n\}_{n=1}^{\infty}$ such that

$$\|\Lambda_n(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|\|_1 \to 0$$

The following lemma is a key tool of the proof of the copmactness of $\mathcal{D}_0(\mathcal{H})$(:(I) in slide11).

Lemma(Asakura)

For any $D \in \mathcal{D}_0(\mathcal{H})$, we can construct the sequence $\{D_n\}_n$ such that

- $D_n \to D$ in WOT.
- D_n can be written as a direct sum of $n \times n$ doubly stochastic and identity operator.

relation to the result of Owari et al(cont.)

- This lemma says

  "anydoubly (sub)stochastic operator (= infinte matrix) D
  can be approximated by doubly stochastic matrix D_n".

- We can construct a LOCC channel Λ_n corresponding to D_n such that $\|\Lambda_n(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|\|_1 \to 0$.