Linear System Games

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Types of non-locality

	Tensor Product	Commuting Operator

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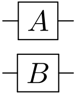
Hilbert Space	$\mathcal{H}_A \otimes \mathcal{H}_B$	
Alice's Operator	$A \otimes \mathbb{1}$	
Bob's Operator	$\mathbb{1} \otimes B$	
Restriction	-	
Circuit		

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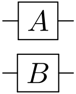
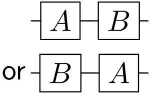
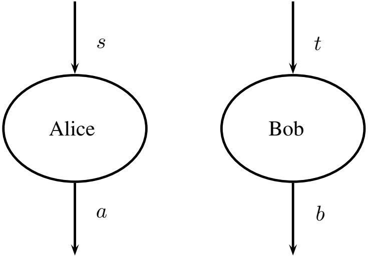
Circuit		

Non-local games



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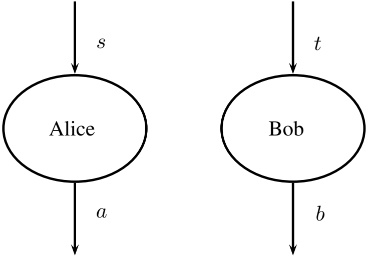

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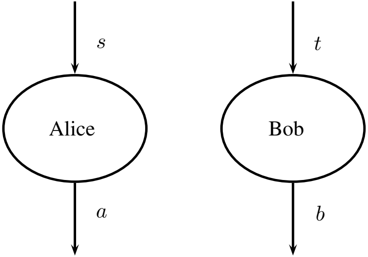

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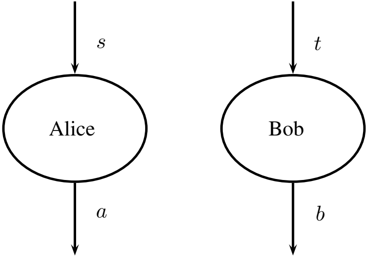

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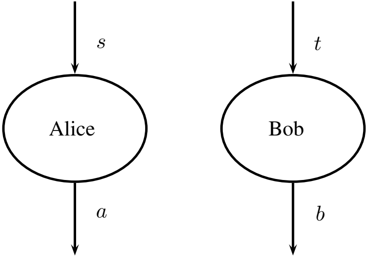

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Example: CHSH game

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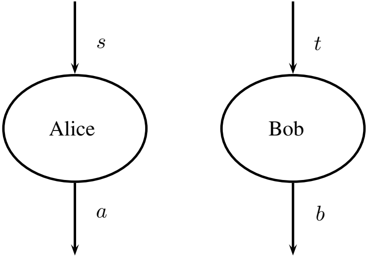

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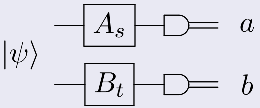
$a, b, s, t \in \{0, 1\}$, winning condition: $a \oplus b = st$
Classical winning Pr: 0.75 (always output 0)
Entanglement winning Pr: $\cos^2 \frac{\pi}{8} \approx 0.85$

Strategies

Tensor product strategies:

- Shared entangled state
  $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
- Local operators
  $A_s \in \mathcal{H}_A$ and $B_t \in \mathcal{H}_B$



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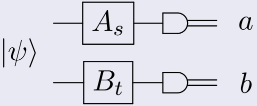

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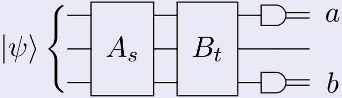

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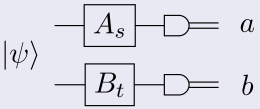

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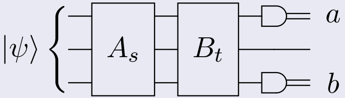

(Classical strategy: no entanglement, no fun.)

Linear System Games

- Non-local game derived from binary[1] linear system $Ax = b$, representing set of linear constraints

[1] Can be easily extended to mod p

Linear System Games

- Non-local game derived from binary[1] linear system $Ax = b$, representing set of linear constraints
  - Example of a logical constraint:
    $x_1 \wedge x_2 \wedge \neg x_3 = 1$
  - Example of a linear constraint:
    $x_1 \oplus x_2 \oplus x_3 = 0$

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- Input:
  - Alice receives row $i \in \{1, \cdots, m\}$, a constraint
  - Bob receives column $j \in \{1, \cdots, n\}$, a variable
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- Winning Output:
  - Alice assigns vector x such that $\sum_j A_{i,j} x_j = b_i$
  - Bob outputs one bit that matches with Alice's x_j.

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Example of linear system game

- Linear system:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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- Inputs: $i = 1$, $j = 1$
  - Alice must output x such that $x_1 \oplus x_2 = 1$
  - If Alice outputs $[1, 0, 0]$, Bob must output 1.
- Perfect classical strategy exists: setting $x = [0, 1, 0]^T$

Perfect strategies

- Classical strategy $\iff$ classical solution

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- Tensor product strategy $\iff$ finite dim. "operator solution" [Cleve & Mittal 2012]

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- Commuting operator strategy $\iff$ infinite dim. operator solution [this paper]

Solutions to linear systems

Classical solution

- $x_j \in \{0, 1\}$
- $\sum_j A_{ij} x_j = b_i$

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Alternative form

- $x_j \in \{-1, 1\}$

$0 \mapsto 1$, $1 \mapsto -1$

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- $x_j \in \{-1, 1\}$
- $\prod_j x_j^{A_{ij}} = (-1)^{b_i}$

$+$ to $\times$

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Turn into matrices

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$$x_1 x_2 = -1 \iff x_2 x_1 = -1$$

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Abstraction

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- $g_j^2 = e,\ J^2 = e$
- $g_j J = J g_j$

J is similar to $-\mathbb{1}$.

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↑ is a representation of ↑ with $J \mapsto -\mathbb{1}$

Summary of types of solutions

	classical sol.	**operator sol.**	**solution group**
binary	$x_j = \pm 1$	$X_j^2 = \mathbb{1}$	$g_j^2 = e,\ J^2 = e$
constraint satisfying	$\prod_j x_j^{A_{ij}} = (-1)^{b_i}$	$\prod_j X_i^{A_{ij}} = (-\mathbb{1})^{b_i}$	$\prod_j g_j^{A_{ij}} = J^{b_i}$
local compatibility	-	$X_j X_{j'} = X_{j'} X_j$ $A_{ij'} = A_{ij'} = 1$	$g_j g_{j'} = g_{j'} g_j$ $A_{ij'} = A_{ij'} = 1$
other	-	-	$[J, g_j] = 1\ \forall g_j$

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2. The linear system has an operator solution (possibly ∞ dimensional)
3. The solution group satisfies $J \neq e$.

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$$1 \subseteq 2 \subseteq 3 \subseteq 4$$

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Strong Tsirelson's Conjecture

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Strong Tsirelson's Conjecture (False [Slofstra 2016])

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Conne's Embedding Conjecture (Open)

Discussions

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- Not clear how to apply group structure for systems other than linear system games.

Q&A

Thank you!

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- Start with a representation of the solution group,
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- Left group multiplication $\rightarrow$ Alice's operators
- Right group multiplication $\rightarrow$ Bob's operators
- $|\psi\rangle = (|e\rangle - |J\rangle)/\sqrt{2}$.