Factoring with Qutrits

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Joint work with Alex Bocharov, Martin Roetteler, Krysta Svore
(QuArC, Microsoft Research)
Why Qutrits?

Qubit = \{ |0\rangle, |1\rangle \} \quad \text{v.s.} \quad \text{Qutrit} = \{ |0\rangle, |1\rangle, |2\rangle \}
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- Topological quantum computation (TQC) by non-abelian anyons. Certain anyon system naturally encodes a qutrit.
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- Experimental implementation, e.g. linear ion traps, cold atoms, entangled photons
- Topological quantum computation (TQC) by non-abelian anyons. Certain anyon system naturally encodes a qutrit. TQC is fault tolerant; has asymptotically better efficiency.
Metaplectic Quantum Computer

SU(2) anyon system (←→ fractional quantum Hall liquids at $\nu = \frac{8}{3}$)

Braiding and measurement $\Rightarrow$ Metaplectic Basis: Qutrit Clifford + 

$|X_i\rangle = |i + 1\rangle$, $|Q_i\rangle = \omega^{\delta i_i} 2|\rangle$;

$H |i\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \omega_{ij} |j\rangle$;

$\sum |i, j\rangle = |i, i+j\rangle$.

$R|2\rangle = \text{diag}(1, 1, -1)$

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Theorem (BCKW, 2015)

Any single qutrit gate can be approximated with precision $\epsilon$ by a metaplectic circuit with $R|2\rangle$-count $O(\log(1/\epsilon))$. (Compared with $O(\log(3^{0.97}(1/\epsilon)))$ Solovay-Kitaev)
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Clifford + $P_9$ Basis

Clifford + $P_9$, $P_9 = \text{diag}(1, \omega_9, \omega_9^2)$, $\omega_9 = e^{\frac{2\pi i}{9}}$. 

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\( P_9 \):
- Qutrit analog of the qubit \( \pi/8 \)-gate.
- \( P_9 \in C_3 \) 3rd Clifford hierarchy.
- Obtained from magic state distillation.
  - magic state \( \mu = |0\rangle + \omega_9|1\rangle + \omega_9^2|2\rangle \).
  - distillation complexity: requires \( O(\log^3(1/\delta)) \) raw magic states to distill one copy of \( \mu \) with fidelity \( 1 - \delta \).
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Theorem (BCRS, 2015, informal)

One can implement ternary arithmetic (e.g. ternary adder, comparison, multiplication, subtraction, etc.) exactly over Clifford + $P_9$ basis.
Embed a qubit \(\{\ket{0}, \ket{1}\}\) in a qutrit \(\{\ket{0}, \ket{1}, \ket{2}\}\).
Emulate Qubits in Qutrit Computer

Embed a qubit \( \{ |0\rangle, |1\rangle \} \) in a qutrit \( \{ |0\rangle, |1\rangle, |2\rangle \} \).

Emulate a qubit gate with a qutrit gate. E.g.,

\[
\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}
\]
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For Shor’s Algorithm, compare the cost in terms of raw magic state count.

\[
\begin{cases} 
\text{Clifford } + \pi/8 \text{(qubit)} & |0\rangle + e^{\pi i/4} |1\rangle \\
\text{Clifford } + R_{|2\rangle} \text{(qutrit)} & |0\rangle + |1\rangle - |2\rangle \\
\text{Clifford } + P_{9} \text{(qutrit)} & |0\rangle + e^{2\pi i/9} |1\rangle + e^{4\pi i/9} |2\rangle 
\end{cases}
\]
Shor’s Factorization–Period Finding

Quantum part:

Given $a < N$, $(a, N) = 1$, find the smallest number $r$, such that $a^r = 1 \mod N$.

1. Prepare quantum state proportional to the following superposition:

$$
\sum_{k=0}^{N^2} |k\rangle|a^k \mod N\rangle
$$

2. Perform quantum Fourier transform of the first register.

3. Measure the first register.
The cost of emulating the binary circuit of period finding is proportional to the cost of emulating the Toffoli gate.

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The Toffoli gate can be emulated exactly in the Clifford $+ P_9$ basis either

1. by a four-qutrit circuit with 6 $P_9$ gates (with one ancilla),
2. or by a three-qutrit circuit with 15 $P_9$ gates (ancilla free).
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The $P_9$ can be approximated by a metaplectic circuit of $R_{|2\rangle\text{-count}}$-

$6 \log_3(1/\epsilon)$. 

Compare the cost of Toffoli gate in qubit/qutrit models:

<table>
<thead>
<tr>
<th>Clean magic states</th>
<th>Raw resources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clifford + $\pi/8$</td>
<td>$7$</td>
</tr>
<tr>
<td>Clifford $^A$ + $P_9$</td>
<td>$15$</td>
</tr>
<tr>
<td>Clifford $^B$ + $P_9$</td>
<td>$6$</td>
</tr>
<tr>
<td>Clifford $^A$ + $R_{</td>
<td>2\rangle}$</td>
</tr>
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</table>

Table "Clifford $^A$" stands for 3-qutrit emulation of the Toffoli gate and "Clifford $^B$" use 4-qutrit emulation with one clean ancilla prepared with SUM gates.
Comparison of cost of Period finding

Compare the cost of implementing period-finding circuit in qubit/qutrit models:

<table>
<thead>
<tr>
<th>Circuits</th>
<th>Online width</th>
<th>Offline width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary QCLA</td>
<td>$3n - w(n)$ (qubits)</td>
<td>$7n(6\log_2(n))^{2.5}$</td>
</tr>
<tr>
<td>Clifford $A + P_9$</td>
<td>$3n - w(n)$ (qutrits)</td>
<td>$15n(3\log_2(n))^3$</td>
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<tr>
<td>Clifford $B + P_9$</td>
<td>$4n - w(n)$ (qutrits)</td>
<td>$6n(3\log_2(n))^3$</td>
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<tr>
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Table($w(n)$ is the Hamming weight of $n$). Clifford $A$ stands for 3-qutrit emulation of the Toffoli gate and case $B$ for the 4-qutrit emulation. The last column in metaplectic rows shown the expected average of the probabilistic width.
Summary & Conclusion

Introduced two qutrit basis:

\[
\begin{align*}
\text{Clifford} + R_{1,2} & \quad \text{Metaplectic TQC} \\
\text{Clifford} + P_9 & \quad \text{Qutrit analog of qubit Clifford} + \frac{\pi}{8}
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Thank you!