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Factoring with Qutrits

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Why Qutrits?

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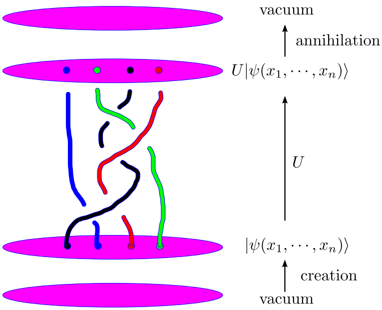
TQC is fault tolerant; has asymptotically better efficiency.

Metaplectic Quantum Computer



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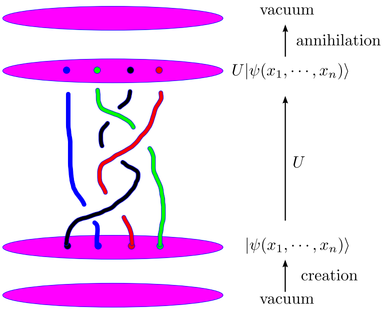

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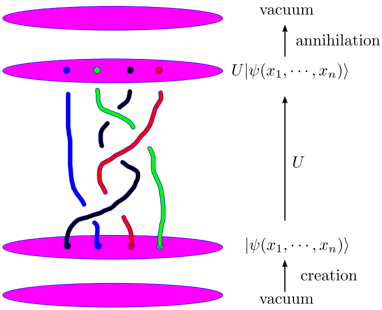
Braiding and measurement
$\Rightarrow$ Metaplectic Basis:
Qutrit Clifford + $R_{|2\rangle}$.

- $X|i\rangle = |i+1\rangle$, $Q|i\rangle = \omega^{\delta_{i,2}}|i\rangle$;
  $H|i\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \omega^{ij}|j\rangle$;
  $\mathrm{SUM}|i,j\rangle = |i, i+j\rangle$.
- $R_{|2\rangle} = \mathrm{diag}(1, 1, -1)$



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Theorem (BCKW, 2015)

Any single qutrit gate can be approximated with precision ϵ by a metaplectic circuit with $R_{|2\rangle}$-count $O(\log(1/\epsilon))$.
(Compared with $O(\log^{3.97}(1/\epsilon))$ Solovay-Kitaev)

Clifford + P_9 Basis

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- Qutrit analog of the qubit $\pi/8$-gate.
- $P_9 \in \mathcal{C}_3$ 3rd Clifford hierarchy.
- Obtained from magic state distillation.
  - magic state $\mu = |0\rangle + \omega_9|1\rangle + \omega_9^2|2\rangle$.
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Theorem (BCRS, 2015, informal)

One can implement ternary arithmetic (e.g. ternary adder, comparison, multiplication, subtraction, etc.) exactly over Clifford + P_9 basis.

Emulate Qubits in Qutrit Computer

Embed a qubit $\{|0\rangle, |1\rangle\}$ in a qutrit $\{|0\rangle, |1\rangle, |2\rangle\}$.

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$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$$

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Emulation efficiency?

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Emulation efficiency?

For Shor's Algorithm, compare the cost in terms of raw magic state count.

$$\begin{cases} \text{Clifford} + \pi/8(\text{qubit}) & |0\rangle + e^{\frac{\pi i}{4}}|1\rangle \\ \text{Clifford} + R_{|2\rangle}(\text{qutrit}) & |0\rangle + |1\rangle - |2\rangle \\ \text{Clifford} + P_9(\text{qutrit}) & |0\rangle + e^{\frac{2\pi i}{9}}|1\rangle + e^{\frac{4\pi i}{9}}|2\rangle \end{cases}$$

Shor's Factorization–Period Finding

Quantum part:

Given $a < N$, $(a, N) = 1$, find the smallest number r, such that $a^r = 1 \mod N$.

1. Prepare quantum state proportional to the following superposition:
$$\sum_{k=0}^{N^2} |k\rangle|a^k \mod N\rangle$$
2. Perform quantum Fourier transform of the first register.
3. Measure the first register.

Qutrit Emulation of Shor's Factorization

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The Toffoli gate can be emulated exactly in the Clifford + P_9 basis either

1. *by a four-qutrit circuit with* **6** *P_9 gates (with one ancilla),*
2. *or by a three-qutrit circuit with* **15** *P_9 gates (ancilla free).*

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Proposition (BRS, 2016)

The P_9 can be approximated by a metaplectic circuit of $R_{|2\rangle}$-count $6\log_3(1/\epsilon)$.

Comparison of cost of implementing Toffoli

Compare the cost of Toffoli gate in qubit/qutrit models:

	Clean magic states	Raw resources
Clifford $+ \pi/8$	7	$7(2 \log_2(1/\delta))^{2.5}$
Clifford $^A + P_9$	15	$15 \log_2^3(1/\delta)$
Clifford $^B + P_9$	6	$6 \log_2^3(1/\delta)$
Clifford $^A + R_{\|2\rangle}$	15	$90 \log_3(1/\delta)$
Clifford $^B + R_{\|2\rangle}$	6	$36 \log_3(1/\delta)$

Table "Clifford ^{A}" stands for 3-qutrit emulation of the Toffoli gate and "Clifford ^{B}" use 4-qutrit emulation with one clean ancilla prepared with SUM gates.

Comparison of cost of Period finding

Compare the cost of implementing period-finding circuit in qubit/qutrit models:

Circuits	Online width	Offline width
Binary QCLA	$3\,n - w(n)$ (qubits)	$7\,n\,(6\,\log_2(n))^{2.5}$
Clifford ^{A} $+$ P_9	$3\,n - w(n)$ (qutrits)	$15\,n\,(3\,\log_2(n))^3$
Clifford ^{B} $+$ P_9	$4\,n - w(n)$ (qutrits)	$6\,n\,(3\,\log_2(n))^3$
Clifford ^{A} $+$ $R_{\|2\rangle}$	$3\,n - w(n)$ (qutrits)	$90 \times 3\,n\,\log_3(n)$
Clifford ^{B} $+$ $R_{\|2\rangle}$	$4\,n - w(n)$ (qutrits)	$36 \times 3\,n\,\log_3(n)$

Table($w(n)$ is the Hamming weight of n). Clifford ^{A} stands for 3-qutrit emulation of the Toffoli gate and case ^{B} for the 4-qutrit emulation. The last column in metaplectic rows shown the expected average of the probabilistic width.

Summary & Conclusion

Introduced two qutrit basis:

$$\begin{cases} \text{Clifford} + R_{|2\rangle} & \text{Metaplectic TQC} \\ \text{Clifford} + P_9 & \text{Qutrit anolog of qubit Clifford} + \pi/8 \end{cases}$$

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Thank you!