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Entanglement-Assisted Quantum Communication
Beating the Quantum Singleton Bound

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Overview

- parameters of quantum error-correcting codes (QECC)
- no-cloning bound
- quantum Singleton bound
- entanglement-assisted QECC
- noiseless teleportation
- noiseless teleportation
- our scheme
- examples
- conclusions

Code Parameters

Quantum Code: $\mathcal{C} = [\![n, k, d]\!]_q$

q: dimension of the subsystems

n: number of subsystems used in total

k: number of (logical) subsystems encoded

d: "minimum distance"

– correct all errors acting on at most $(d-1)/2$ subsystems

– detect all errors acting on less than d subsystems

– correct all errors at less than d *known* positions ("erasures")

trivial absolute bound: $k \leq n - (d-1) = n + 1 - d$

equivalently: $\boxed{d \leq n + 1 - k}$

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No-Cloning Bound

Assumption: $\mathcal{C} = [\![n, 1, n/2 + 1]\!]_q$ exists

encoded state: $\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$

splitting:	$\sum_i \alpha_i^2 |\psi_i\rangle\langle\psi_i|$	$\sum_i \alpha_i^2 |\phi_i\rangle\langle\phi_i|$
	$\Downarrow$	$\Downarrow$
padding:	$\left(\sum_i \alpha_i^2 |\psi_i\rangle\langle\psi_i|\right) \otimes \left(|0\rangle\langle 0|\right)^{\otimes n/2}$	$\left(|0\rangle\langle 0|\right)^{\otimes n/2} \otimes \left(\sum_i \alpha_i^2 |\phi_i\rangle\langle\phi_i|\right)$
	$\Downarrow$	$\Downarrow$
correction:	$\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$	$\sum_i \alpha_i |\psi_i\rangle|\phi_i\rangle$

two independent copies

$\Longrightarrow$ no-cloning bound: $d - 1 < n/2$ or $\boxed{2d < n + 2}$

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Quantum Singleton Bound

[E. Rains, Nonbinary Quantum Codes, quant-ph/9703048]

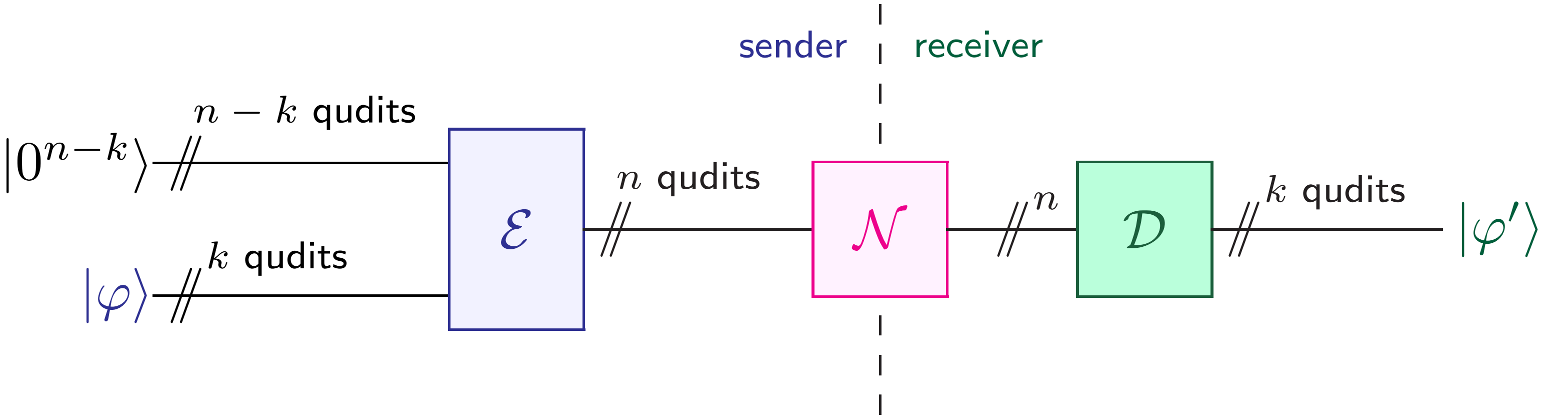
Let $\mathcal{C} = [\![n, k, d]\!]_q$ be a quantum error-correcting code. Then

$$2d \leq n + 2 - k.$$

Proof (sketch:) [J. Preskill, Lecture Notes, Sect. 7.8.3]

- splitting of the code into $(d-1) + (d-1) + (n+2-2d)$ qudits
- maximally entangled state of input and reference system R
- the first two blocks with $d-1$ qudits have no correlations with R
- the entropy of the third block with $n+2-2d$ qudits is at least $k \log q$

Quantum Error-Correcting Code (QEC)

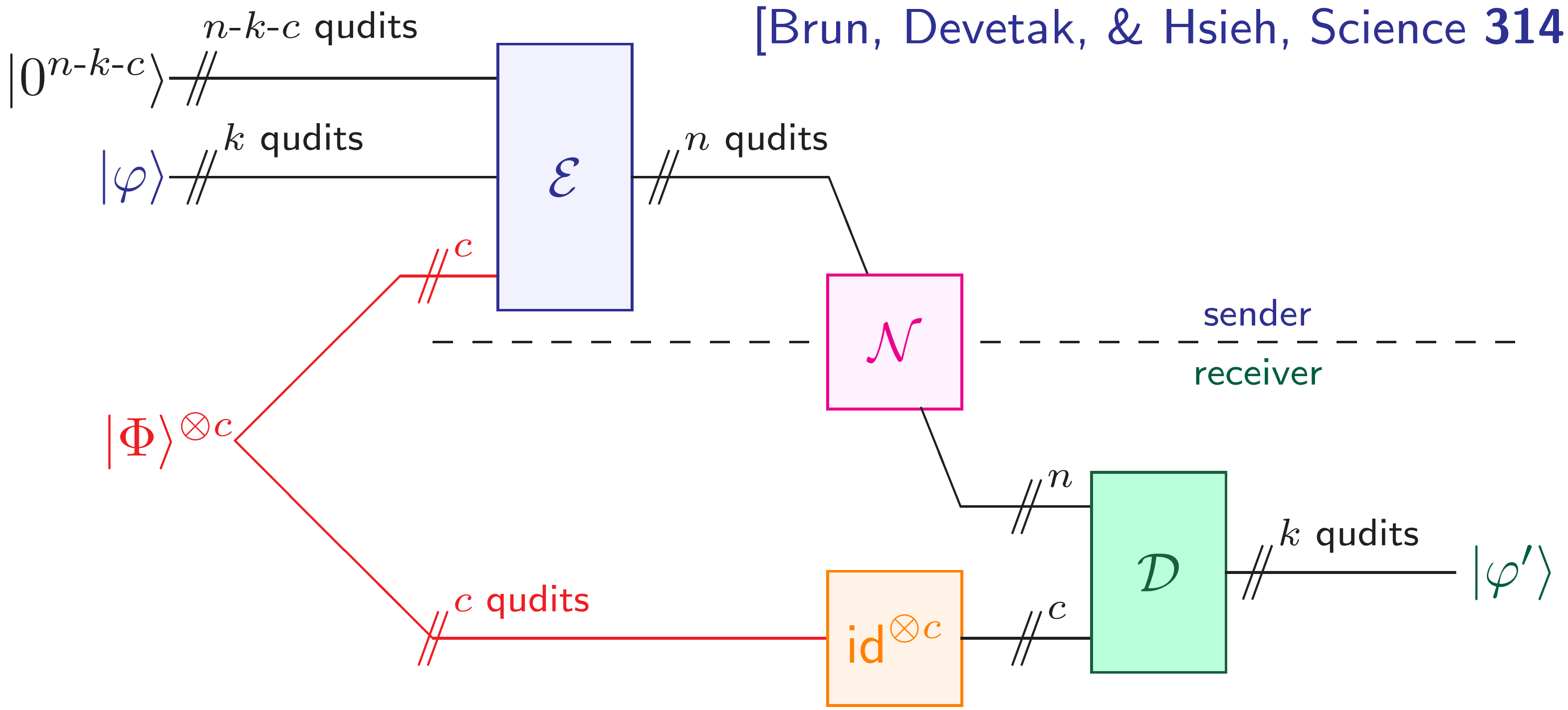


Scheme of a communication protocol using a QECC $[\![n,k,d]\!]_q$

quantum Singleton bound: $\boxed{2d \le n+2-k}$

Entanglement-Assisted QECC

[Brun, Devetak, & Hsieh, Science **314**, 436 (2006)]



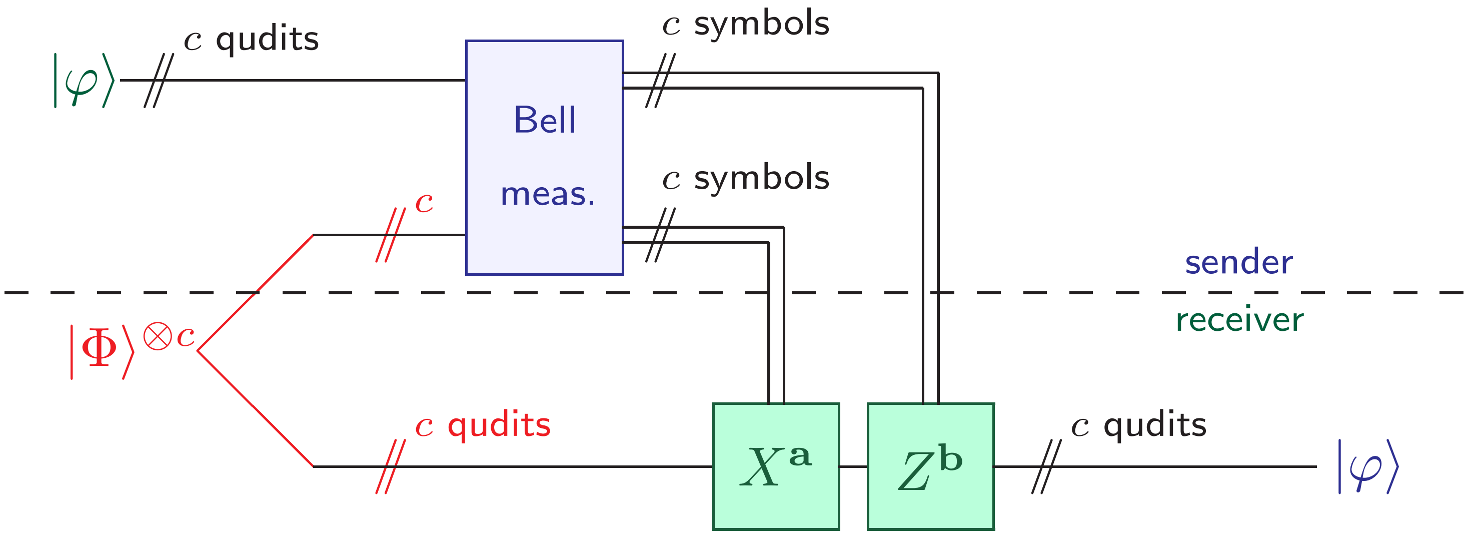
Communication scheme using an entanglement-assisted QECC $[\![n, k, d; c]\!]_q$.

similar to a QECC of length $n + c$

$\Longrightarrow$ **conjectured bound:** $\boxed{2d \leq n + c + 2 - k}$

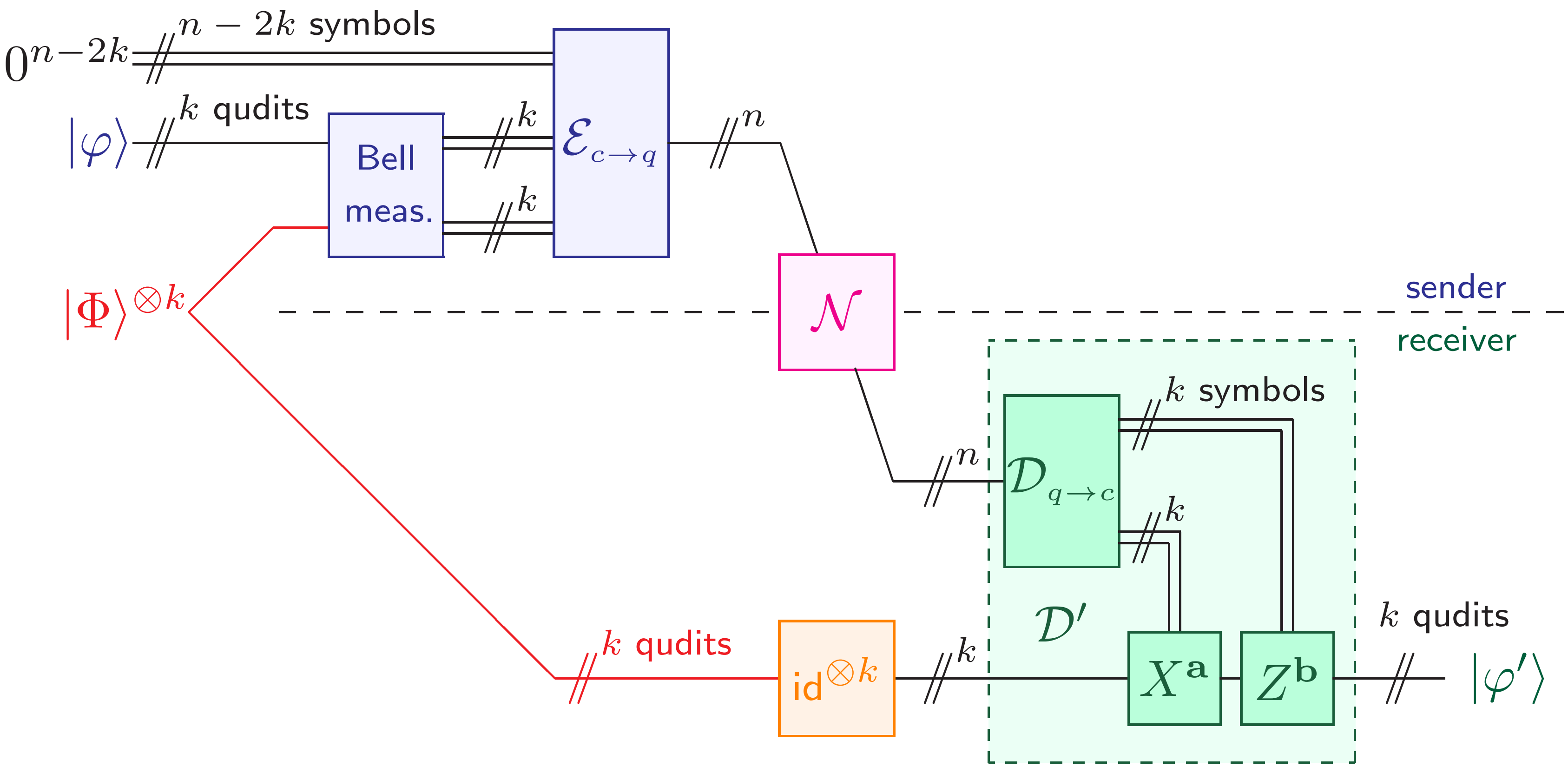
[Brun, Devetak, & Hsieh, arXiv:quant-ph/0608027]

Noiseless Teleportation



The maximally entangled states $|\Phi\rangle^{\otimes c}$ can be prepared by either party, or even a third party.

The New Scheme



Teleportation-based scheme using $c = k$ maximally entangled states.

bound: $d \leq n + 1 - 2k$

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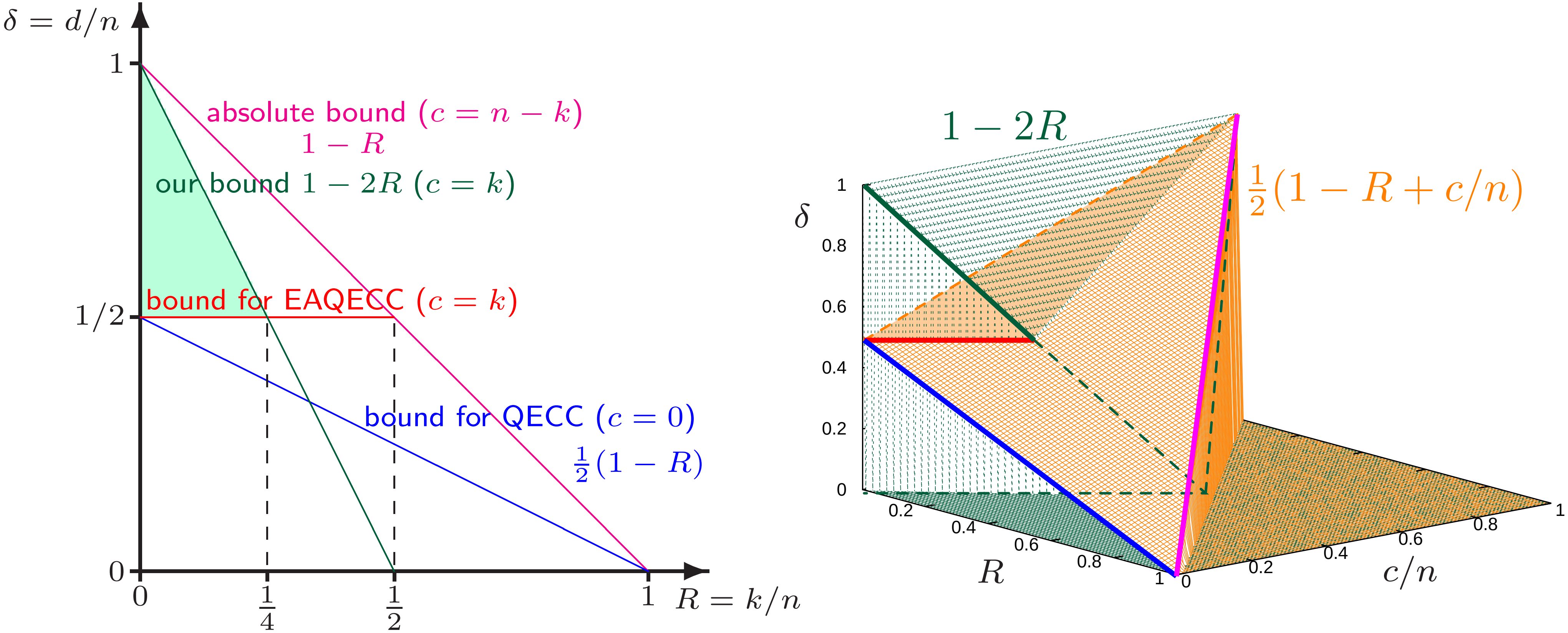
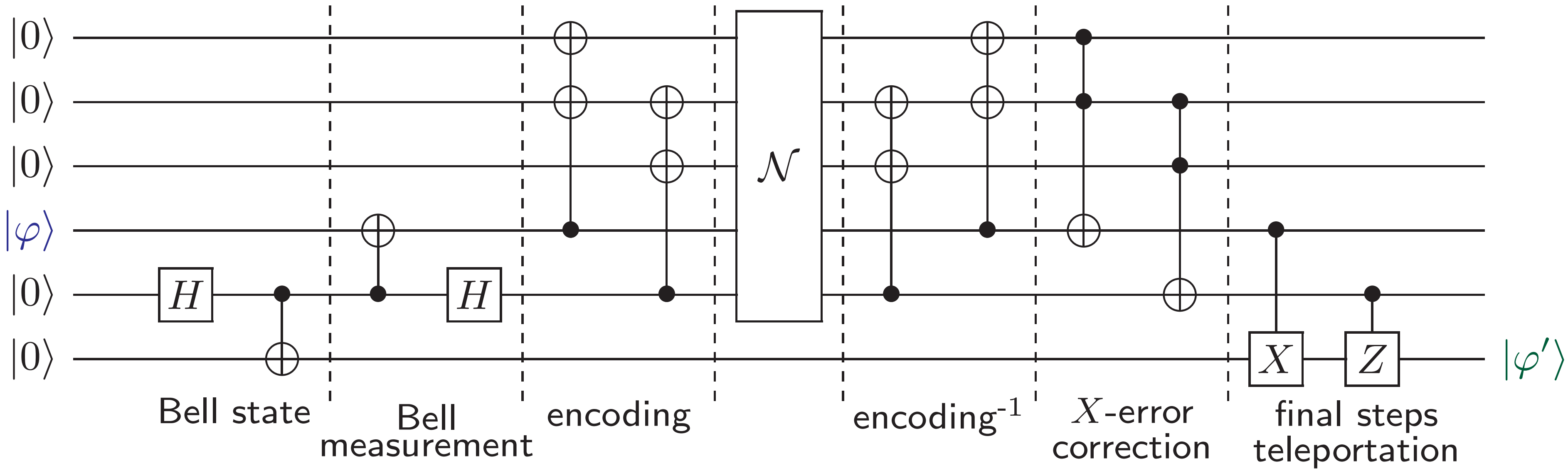
Bounds



Asymptotic bounds (length $n \to \infty$) on the normalized minimum distance $\delta = d/n$ as a function of the code rate $R = k/n$.

Example: $[\![5,1,3;1]\!]_2$



- classical code $[5,2,3]_2$ generated by $G = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
- standard QECC $[\![5,1,3]\!]_2$ exists, but our scheme additionally tolerates all phase errors

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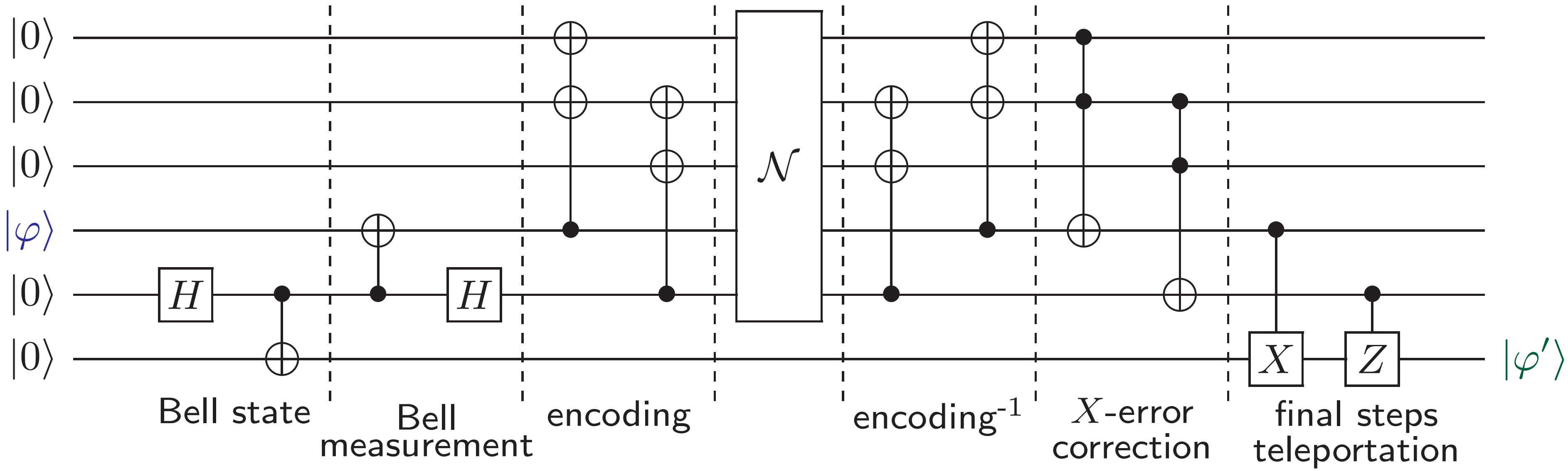
Example: $[\![5,1,3;1]\!]_2$



stabilizer & normalizer:

$$\left(\begin{array}{ccccc|c} Z & I & I & I & I & I \\ I & Z & I & I & I & I \\ I & I & Z & I & I & I \\ I & I & I & I & Z & I \\ I & I & I & I & I & Z \\ \hline I & I & I & X & I & I \\ I & I & I & Z & I & I \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} Z & I & I & Z & I & I \\ I & Z & I & Z & Z & I \\ I & I & Z & I & Z & I \\ X & X & I & X & Z & X \\ I & X & X & I & X & Z \\ \hline X & X & I & X & I & I \\ I & X & X & Z & X & I \end{array}\right)$$

Example: $[\![5,1,3;1]\!]_2$

stabilizer & normalizer:

$$\left(\begin{array}{ccccc|c} Z & I & I & I & I & I\\ I & Z & I & I & I & I\\ I & I & Z & I & I & I\\ I & I & I & I & Z & I\\ I & I & I & I & I & Z\\ \hline I & I & I & X & I & I\\ I & I & I & Z & I & I \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} Z & I & I & Z & I & I\\ I & Z & I & Z & Z & I\\ I & I & Z & I & Z & I\\ X & X & I & X & Z & X\\ I & X & X & I & X & Z\\ \hline X & X & I & X & I & I\\ I & X & X & Z & X & I \end{array}\right)$$

split weight enumerators [C.-Y. Lai & A. Ashikhmin, arXiv:1602.00413v2]

$$W_{\mathcal{C}}^{(5,1)}(Y;V) = (1+2Y^2+4Y^3+Y^4)+(4Y^3+12Y^4+8Y^5)V$$

$$W_{\mathcal{C}^\perp}^{(5,1)}(Y;V) = (1+2Y^2+8Y^3+13Y^4+8Y^5) + (5Y+8Y^2+18Y^3+40Y^4+25Y^5)V$$

$$W_{\mathcal{C}^\perp}^{(5,1)}(Y;V) - W_{\mathcal{C}}^{(5,1)}(Y;V) = (4Y^3+12Y^4+8Y^5) + (5Y+8Y^2+14Y^3+28Y^4+17Y^5)V$$

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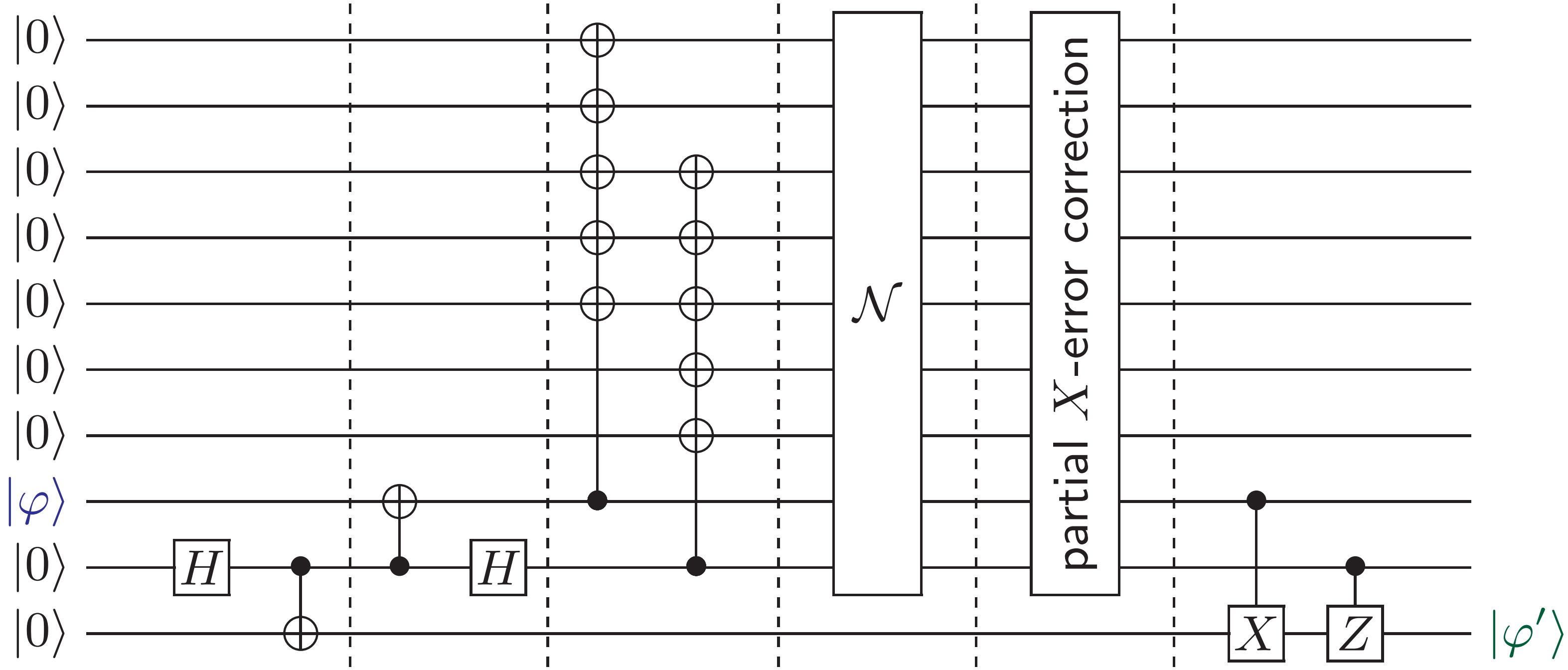
Example: $[\![9,1,6;1]\!]_2$



- classical code $[9,2,6]_2$
- original bound for $n=9$, $k=1$, $c=1$: $\boxed{2d \le n+c+2-k = 11}$

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Example: $[\![9,1,6;1]\!]_2$

stabilizer & normalizer:

$$\left(\begin{array}{ccccccccc|c}
Z & I & I & I & I & I & I & Z & I & I\\
I & Z & I & I & I & I & I & Z & I & I\\
I & I & Z & I & I & I & I & Z & Z & I\\
I & I & I & Z & I & I & I & Z & Z & I\\
I & I & I & I & Z & I & I & Z & Z & I\\
I & I & I & I & I & Z & I & I & Z & I\\
I & I & I & I & I & I & Z & I & Z & I\\
X & X & X & X & X & I & I & X & Z & X\\
I & I & X & X & X & X & X & I & X & Z\\
\hline
X & X & X & X & X & I & I & X & I & I\\
I & I & X & X & X & X & X & Z & X & I
\end{array}\right)$$

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Example: $[\![9,1,6;1]\!]_2$

split weight enumerators [C.-Y. Lai & A. Ashikhmin, arXiv:1602.00413v2]

$$W_{\mathcal{C}}^{(9,1)}(Y;V) = (1 + 9Y^2 + 27Y^3 + 27Y^4 + 27Y^5 + 27Y^6 + 9Y^7 + Y^9) + (48Y^6 + 144Y^7 + 144Y^8 + 48Y^9)V$$

$$W_{\mathcal{C}^\perp}^{(9,1)}(Y;V) = (1 + 9Y^2 + 27Y^3 + 27Y^4 + 27Y^5 + 75Y^6 + 153Y^7 + 144Y^8 + 49Y^9) + (9Y + 27Y^2 + 57Y^3 + 99Y^4 + 99Y^5 + 201Y^6 + 459Y^7 + 441Y^8 + 144Y^9)V$$

$$W_{\mathcal{C}^\perp}^{(9,1)}(Y;V) - W_{\mathcal{C}}^{(9,1)}(Y;V) = (48Y^6 + 144Y^7 + 144Y^8 + 48Y^9) + (9Y + 27Y^2 + 57Y^3 + 99Y^4 + 99Y^5 + 153Y^6 + 315Y^7 + 297Y^8 + 96Y^9)V$$

Conclusions

Summary

- using a simple teleporation-based scheme beats the quantum Singleton bound
- the scheme uses the same assumptions & operations as "standard" entangelement-assisted quantum codes

Outlook

- new schemes to get more flexibility how much entanglemnt is used
- derive new upper bounds, e.g., using linear programming techniques