Separability of Bosonic Systems

Nengkun Yu

University of Technology Sydney,
University of Waterloo,
University of Guelph

August 30, 2016

Working Object: Quantum states

- Pure state: unit vector of Hilbert space.

  Mixed state(density matrix) of $\mathcal{H}$: semi-definite matrix $\rho \in D(\mathcal{H})$,

$$\mathrm{tr}(\rho) = 1,$$
$$\rho \geq 0.$$

- Multipartite Hilbert space is defined as the tensor product of Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_k$$

  $\mathcal{H}$ with basis $|\alpha_{1,j_1}\rangle|\alpha_{2,j_2}\rangle \cdots |\alpha_{k,j_k}\rangle$, where $|\alpha_{t,j_t}\rangle$ is basis of $\mathcal{H}_t$.

  Not all state in $\mathcal{H}$ can be written as the product form $|\alpha_{1,j_1}\rangle|\alpha_{2,j_2}\rangle \cdots |\alpha_{k,j_k}\rangle$.

  Most of them can not be written in that way.—Entanglement!!!

Working Object: Quantum states

- Pure state: unit vector of Hilbert space.

  Mixed state(density matrix) of $\mathcal{H}$: semi-definite matrix $\rho \in D(\mathcal{H})$,

$$\begin{aligned} \mathrm{tr}(\rho) &= 1, \\ \rho &\geq 0. \end{aligned}$$

- Multipartite Hilbert space is defined as the tensor product of Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_k$$

  $\mathcal{H}$ with basis $|\alpha_{1,j_1}\rangle|\alpha_{2,j_2}\rangle \cdots |\alpha_{k,j_k}\rangle$, where $|\alpha_{t,j_t}\rangle$ is basis of $\mathcal{H}_t$.

  Not all state in $\mathcal{H}$ can be written as the product form $|\alpha_{1,j_1}\rangle|\alpha_{2,j_2}\rangle \cdots |\alpha_{k,j_k}\rangle$.

  Most of them can not be written in that way.—Entanglement!!!

- For pure state, it is entangled iff it is not the product form.

  For mixed state, it is entangled iff it is not separable.—define it latter.

- Fundamental feature in quantum physics.

- Resource of quantum information processing: Superdense coding, teleportation, graph state–measurement based quantum computing.

  Quantum computing: provable secure information transformation–BB84, faster computing speed–Shor's factoring algorithm, Grover search.

- For pure state, it is entangled iff it is not the product form.

  For mixed state, it is entangled iff it is not separable.—define it latter.

- Fundamental feature in quantum physics.

- Resource of quantum information processing: Superdense coding, teleportation, graph state–measurement based quantum computing.

  Quantum computing: provable secure information transformation–BB84, faster computing speed–Shor's factoring algorithm, Grover search.

Working Property: Entanglement

- For pure state, it is entangled iff it is not the product form.

  For mixed state, it is entangled iff it is not separable.—define it latter.

- Fundamental feature in quantum physics.

- Resource of quantum information processing: Superdense coding, teleportation, graph state–measurement based quantum computing.

  Quantum computing: provable secure information transformation–BB84, faster computing speed–Shor's factoring algorithm, Grover search.

- We shall study the following *separability problem* for bosonic system

  For a given quantum states ρ, determine whether it is separable, *i.e.*,

$$\rho = \sum_j \bigotimes_{k=1}^{N} |\alpha_{j_k}\rangle\langle\alpha_{j_k}|.$$

- An n-qubit state ρ is called bosonic if

$$\rho = F_{ij}\rho = \rho F_{ij},$$

  where $F_{i,j}$ is the SWAP operator on the i-th and j-th qubit system.

- We shall study the following *separability problem* for bosonic system

  For a given quantum states ρ, determine whether it is separable, *i.e.*,

$$\rho = \sum_j \bigotimes_{k=1}^{N} |\alpha_{j_k}\rangle\langle\alpha_{j_k}|.$$

- An n-qubit state ρ is called bosonic if

$$\rho = F_{ij}\rho = \rho F_{ij},$$

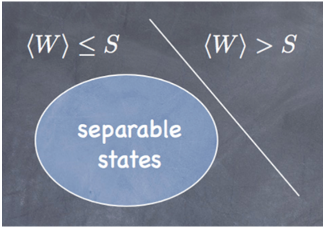
  where $F_{i,j}$ is the SWAP operator on the i-th and j-th qubit system.

Motivation 1: Quantifying entanglement

Definition of entanglement.

A state is entangled iff it is not separable.

Entanglement witnesses.



The relation between positive partial transpose (PPT) and entanglement.

- Γ means the partial transpose

$$(|ij\rangle\langle kl|)^{\Gamma} = |kj\rangle\langle il|.$$

A quantum state ρ_{AB} is separable only if $\rho^{\Gamma} \geq 0$, it is PPT. A Peres, PRL 1996.

- For $2 \otimes 2$ and $2 \otimes 3$ states, it is separable iff it is PPT.
M. Horodecki, P. Horodecki, and R. Horodecki, PLA 1998.
In general, separability problem is NP-Hard—L. Gurvits 2004.
Characterization of PPT entanglement is an important problem.

The relation between positive partial transpose (PPT) and entanglement.

- Γ means the partial transpose

$$(|ij\rangle\langle kl|)^{\Gamma} = |kj\rangle\langle il|.$$

  A quantum state ρ_{AB} is separable only if $\rho^{\Gamma} \geq 0$, it is PPT. A Peres, PRL 1996.

- For $2 \otimes 2$ and $2 \otimes 3$ states, it is separable iff it is PPT.
  M. Horodecki, P. Horodecki, and R. Horodecki, PLA 1998.
  In general, separability problem is NP-Hard—L. Gurvits 2004.
  Characterization of PPT entanglement is an important problem.

Motivation of this work

- Dramatic difference between multipartite entanglement and bipartite entanglement.

  There exist tripartite entangled states, which has no bipartite entanglement, even in three-qubit system, see the following

  $$\rho = \frac{1}{4}(I - \sum_i |UPB_i\rangle\langle UPB_i|),$$

  $|UPB_i\rangle$ is UPBs in three-qubit system.

- Bosonic system, one of the most important multi-qubit system, "Bose-Einstein statistics"...

  Its entanglement property is not clear yet, the relation between entanglement property and symmetry.

Motivation of this work

- Dramatic difference between multipartite entanglement and bipartite entanglement.

  There exist tripartite entangled states, which has no bipartite entanglement, even in three-qubit system, see the following

  $$\rho = \frac{1}{4}(I - \sum_i |UPB_i\rangle\langle UPB_i|),$$

  $|UPB_i\rangle$ is UPBs in three-qubit system.

- Bosonic system, one of the most important multi-qubit system, "Bose-Einstein statistics"...
  Its entanglement property is not clear yet, the relation between entanglement property and symmetry.

- For a given n-qubit bosonic state ρ, *i.e.*, $\rho = F_{i,j}\rho = \rho F_{i,j}$, determine whether ρ is separable, that is

$$\rho = \sum_j \bigotimes_{k=1}^{N} |\alpha_{j_k}\rangle\langle\alpha_{j_k}|,$$

Observe that this is equivalent to determine whether

$$\rho = \sum_j \alpha_j^{\otimes N}.$$

- In particular, we are interested in the separability of the general diagonal symmetric states, GDS,

$$\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|,$$

where N-qubit Dicke states are defined as

$$|D_{N,n}\rangle := P_{\mathrm{sym}}\left(|0\rangle^{\otimes n} \otimes |1\rangle^{\otimes N-n}\right),$$

Working problem: The separability of n-qubit bosonic states

- For a given n-qubit bosonic state ρ, *i.e.*, $\rho = F_{i,j}\rho = \rho F_{i,j}$, determine whether ρ is separable, that is

$$\rho = \sum_j \bigotimes_{k=1}^{N} |\alpha_{j_k}\rangle\langle\alpha_{j_k}|,$$

  Observe that this is equivalent to determine whether

$$\rho = \sum_j \alpha_j^{\otimes N}.$$

- In particular, we are interested in the separability of the general diagonal symmetric states, GDS,

$$\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|,$$

  where N-qubit Dicke states are defined as

$$|D_{N,n}\rangle := P_{\mathrm{sym}}\left(|0\rangle^{\otimes n} \otimes |1\rangle^{\otimes N-n}\right),$$

What have been done so far?

- K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein Annals of Physics (2002). No PPT entanglement in three-qubit bosonic system.
- G. Toth and O. Ghne, PRL 2009. Bipartite symmetric systems .
- G. Toth and O. Ghne, Applied Physics B 98.617 (2010). Bipartite symmetric systems.
- J. Tura, R. Augusiak, P. Hyllus, M. Kus, J. Samsonowicz, and M. Lewenstein, PRA, 2012. Four-qubit entangled symmetric states with positive partial transpositions.
- E. Wolfe, S.F. Yelin, PRL(2014). Numerical result for GDS.

Complete characterization of separability of GDS

Theorem

The GDS state $\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|$ *is separable if* M_0, M_1 *are positive semi-definite, i.e.,*

$$M_0 := \begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{m_0} \\ \chi_1 & \chi_2 & \cdots & \chi_{m_0+1} \\ \cdots & \cdots & \cdots & \cdots \\ \chi_{m_0} & \chi_{m_0+1} & \cdots & \chi_{2m_0} \end{pmatrix} \geq 0, \quad (1)$$

$$M_1 := \begin{pmatrix} \chi_1 & \chi_2 & \cdots & \chi_{m_1} \\ \chi_2 & \chi_3 & \cdots & \chi_{m_1+1} \\ \cdots & \cdots & \cdots & \cdots \\ \chi_{m_1} & \chi_{m_1+1} & \cdots & \chi_{2m_1-1} \end{pmatrix} \geq 0, \quad (2)$$

where $m_0 := [\frac{N}{2}]$ *and* $m_1 := [\frac{N+1}{2}]$.

- We need the following

Lemma

A general diagonal symmetric state ρ is separable if $\mathrm{tr}(W_0\rho) \leq 0$ for all entanglement witness W_0 with form $\sum_i^N m_i|D_{N,i}\rangle\langle D_{N,i}|$. That is $\mathrm{tr}(W_0\rho) \leq 0$ holds for all W_0 which enjoys the property that $\mathrm{tr}(W_0\alpha^{\otimes N}) \leq 0$ for qubit α.

- W_0 is a entanglement witness if and only if $\sum_{k=0}^N m_k|z|^{2k}$ is always non-positive for all $z \in \mathbb{C}$,

$$g(r) \leq 0 \text{ for all } r \geq 0,$$

for real coefficient polynomial $g(x) := \sum_{k=0}^N m_k x^k$, whose value $g(x)$ is always non-positive for non-negative x.

- We need the following

Lemma

A general diagonal symmetric state ρ is separable if $\mathrm{tr}(W_0\rho) \leq 0$ for all entanglement witness W_0 with form $\sum_i^N m_i|D_{N,i}\rangle\langle D_{N,i}|$. That is $\mathrm{tr}(W_0\rho) \leq 0$ holds for all W_0 which enjoys the property that $\mathrm{tr}(W_0\alpha^{\otimes N}) \leq 0$ for qubit α.

- W_0 is a entanglement witness if and only if $\sum_{k=0}^N m_k|z|^{2k}$ is always non-positive for all $z \in \mathbb{C}$,

$$g(r) \leq 0 \text{ for all } r \geq 0,$$

for real coefficient polynomial $g(x) := \sum_{k=0}^N m_k x^k$, whose value $g(x)$ is always non-positive for non-negative x.

- Characterization of such polynomials is given

Lemma

A real coefficient polynomial $g(x)$ satisfies that $g(r) \leq 0$ for all $r \geq 0$ if there exist real coefficient polynomial $P_i(x), Q_i(x)$ such that

$$g(x) = -\sum_i xP_i^2(x) - \sum_i Q_i^2(x).$$

- Use the fundamental theorem of algebra,

$$g(x) = a_0 \prod (x - z_k)^{l_k}.$$

For non real root z_k, we know that for all real r,

$$(r - z_k)(r - \bar{z}_k) = (r - Re(z_k))^2 + Im^2(z_k) \geq 0.$$

For non-positive z_k, we know that for all $r \geq 0$,

$$r - z_k = r + (-z_k) \geq 0.$$

For positive z_k, its power l_k must be even.
Thus, expanding $g(x) = a_0 \prod (x - z_k)^{l_k}$ confirms the only if part.

- Characterization of such polynomials is given

Lemma

A real coefficient polynomial $g(x)$ satisfies that $g(r) \leq 0$ for all $r \geq 0$ if there exist real coefficient polynomial $P_i(x), Q_i(x)$ such that

$$g(x) = -\sum_i xP_i^2(x) - \sum_i Q_i^2(x).$$

- Use the fundamental theorem of algebra,

$$g(x) = a_0 \prod (x - z_k)^{l_k}.$$

For non real root z_k, we know that for all real r,

$$(r - z_k)(r - \bar{z}_k) = (r - Re(z_k))^2 + Im^2(z_k) \geq 0.$$

For non-positive z_k, we know that for all $r \geq 0$,

$$r - z_k = r + (-z_k) \geq 0.$$

For positive z_k, its power l_k must be even.
Thus, expanding $g(x) = a_0 \prod (x - z_k)^{l_k}$ confirms the only if part.

Theorem

The GDS state $\rho = \sum_{n=0}^{N} \chi_n |D_{N,n}\rangle\langle D_{N,n}|$ is separable if it is PPT. More precisely, iff it is PPT under the partial transpose of $m_0 = [\frac{N}{2}]$ subsystems.

If part is easy.

Only if part is proved by the following,

$$PPT \Rightarrow M_0, M_1 \geq 0 \Rightarrow Separability.$$

Multipartite entanglement exists iff bipartite entanglement exists for GDS.

$d \otimes d$ GDS state is defined as $\rho = \sum_{i,j=1}^{d} \chi_{i,j} |\psi_{i,j}\rangle\langle\psi_{i,j}|$, with $|\psi_{i,j}\rangle := \begin{cases} |ii\rangle & \text{if } i = j, \\ |ij\rangle + |ji\rangle & \text{otherwise.} \end{cases}$ being some basis of the bosonic subspace of $d \otimes d$ system, *i.e.*, the symmetric subspace.

Then, we can obtain that,

Theorem

It is NP-Hard to decide whether a given $d \otimes d$ GDS state is separable. On the other hand, $\rho = \sum_{i,j=1}^{d} \chi_{i,j} |\psi_{i,j}\rangle\langle\psi_{i,j}|$ is separable iff $\chi = (\chi_{ij})_{d \times d}$ is semi-definite positive for $d = 3, 4$.

Notice that $\rho = (U \otimes U)\rho(U \otimes U)^\dagger$ holds for all diagonal qudit unitary U. Then, ρ is separable iff there exist qudit states $|\alpha_k\rangle = \sum_{j=1}^{d} x_{k,j}|j\rangle$ such that

$$\begin{aligned}
\rho &= \sum_k \int (U \otimes U)\alpha_k^{\otimes 2}(U \otimes U)^\dagger dU \\
&= \sum_k |x_{k,i}|^2 |x_{k,j}|^2 |\psi_{i,j}\rangle\langle\psi_{i,j}|. \\
\Leftrightarrow \chi : &= (\chi_{ij})_{d \times d} = \sum_k \vec{x_k}\vec{x_k}^T.
\end{aligned}$$

where dU ranging over all diagonal unitaries, and $\vec{x_k} = (|x_{k,1}|^2, \cdots, |x_{k,d}|^2)^T \in \mathbb{R}^d$.

Recall that the cone of complete completely positive matrices is define as

$$\mathcal{C} = \{\sum_i \vec{y_k}\vec{y_k}^T : \vec{y_k} \in \mathbb{R}_+^d\},$$

where $\mathbb{R}_+^d$ stands for the d-dimensional vector space whose entries are all non-negative.

The decision problem on checking the completely positivity of given matrix is NP-Hard for general d while for $d = 3, 4$, checking that the matrix is positive semidefinite and has all entries ≥ 0 is both necessary and sufficient.

Thank you!