Geometric approach to entanglement quantification with polynomial measures

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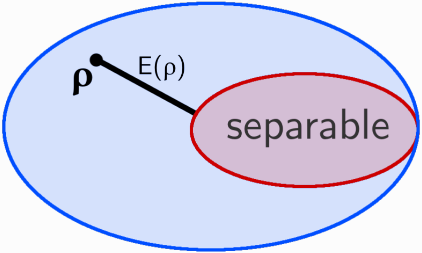
BR and G. Adesso, Phys. Rev. A **94**, 022324 (2016)

BR and G. Adesso, Phys. Rev. Lett. **116**, 070504 (2016)

Quantifying entanglement

Entanglement measures

- Geometric (distance-based)



- Algebraic

polynomial measures (concurrence, three-tangle, ...),
operational measures (distillable ent., entanglement cost, ...),
...

Entanglement measures

$E(|\psi\rangle)$ is an entanglement measure if

- $E(|\psi\rangle) = 0$ if $|\psi\rangle$ is separable
- E is invariant under local unitary transformations
- E is an **entanglement monotone**:
  does not increase on average under Stochastic Local Operations and Classical Communication (SLOCC)

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$$E\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) \leq \sum_i p_i E(|\psi_i\rangle)$$

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$$E\left(\sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|\right) \leq \sum_i p_i\, E\left(\left|\psi_i\right\rangle\right)$$

Extension to mixed states: the **convex roof**

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i\, E(|\psi_i\rangle) \quad \text{with} \quad \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| = \rho$$

A. Uhlmann, Open Sys. & Inf. Dyn. **5**, 209 (1998)

G. Vidal, J. Mod. Opt. **47**, 355 (2000)

Polynomial measures

Polynomial invariant of homogeneous degree d:

$$P_d\big(c\,L\,|\psi\rangle\,\big) = c^d\,P_d\big(\,|\psi\rangle\,\big)$$

with $L \in SL(2,\mathbb{C})^{\otimes n}$, $c > 0$.

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Measure of entanglement: $|P_d(|\psi\rangle)| \to |P_d(\rho)|$

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Measure of entanglement: $|P_d(|\psi\rangle)| \to |P_d(\rho)|$

Examples of polynomial measures:

- concurrence C ($n = 2$, $d = 2$)

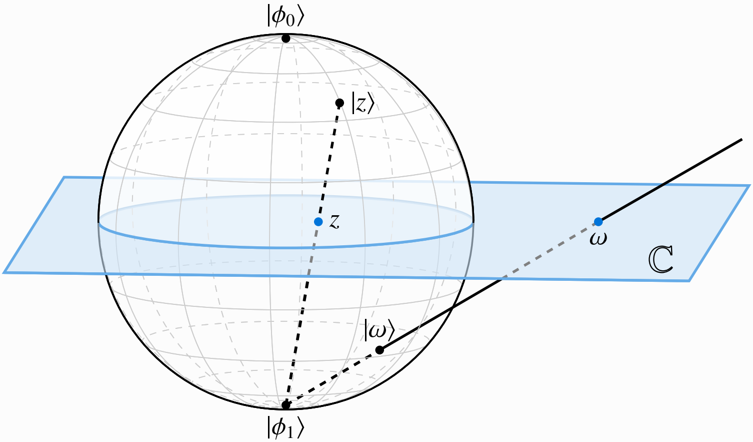
  S. Hill and W. K. Wootters, PRL **78**, 5022 (1997)

- three-tangle τ ($n = 3$, $d = 4$)

  V. Coffman, J. Kundu, and W. K. Wootters, PRA **61**, 052306 (2000)

Geometry of entanglement

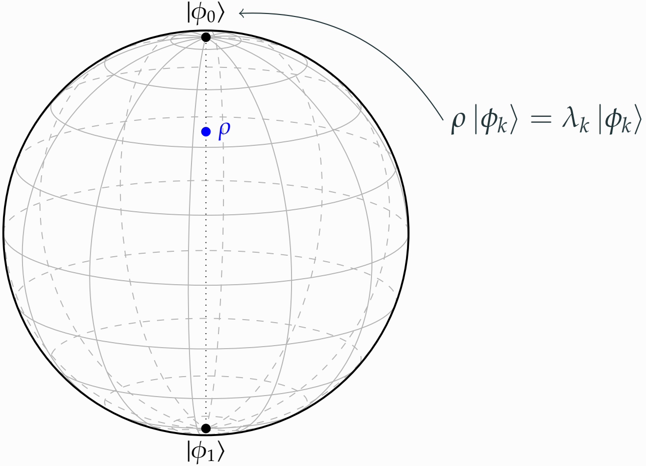
The Bloch sphere as a stereographic projection



$$|z\rangle = |\phi_0\rangle + z\,|\phi_1\rangle$$

$$|\omega\rangle = |\phi_0\rangle + \omega\,|\phi_1\rangle$$

Zero polytope

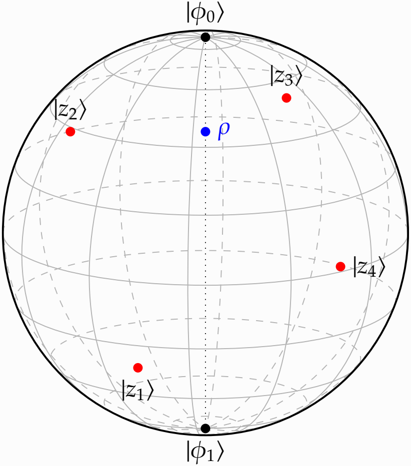


$$E(|z\rangle) = E\Big(\; |\phi_0\rangle + z\,|\phi_1\rangle \; \Big) = 0$$

ρ is rank 2

degree 4

Zero polytope

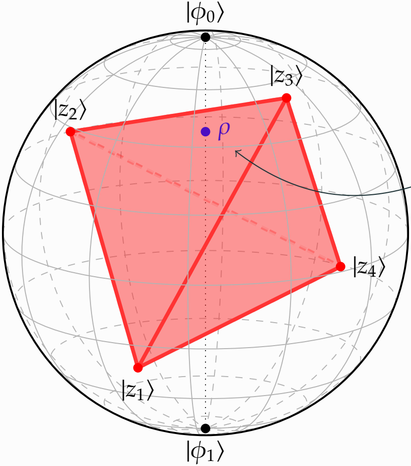


$$E(|z\rangle) = E\Big(\; |\phi_0\rangle + z\,|\phi_1\rangle \; \Big) = 0$$

degree 4 → E

rank 2 → $|\phi_0\rangle + z\,|\phi_1\rangle$

Zero polytope



$\rho \in$ zero polytope

$\Rightarrow E(\rho) = 0$

$$E(|z\rangle) = E\Big(\; |\phi_0\rangle + z\,|\phi_1\rangle \; \Big) = 0$$

degree 4 rank 2

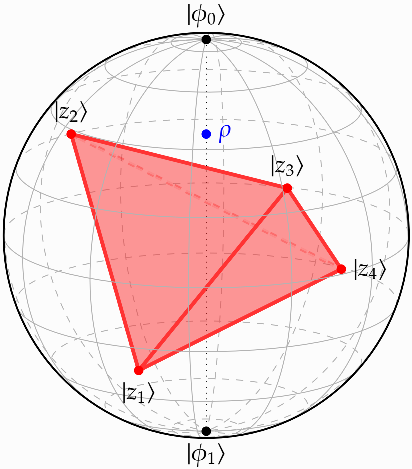
Zero polytope

R. Lohmayer et al.,

PRL **97**, 260502 (2006)

A. Osterloh et al.,

PRA **77**, 032310 (2008)

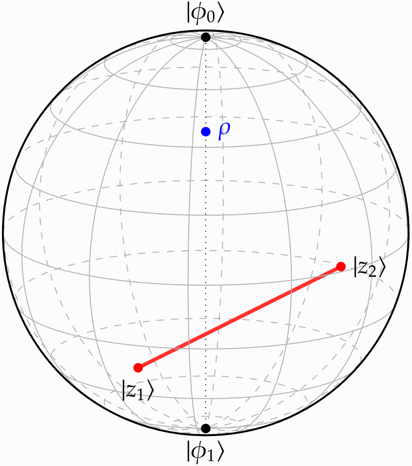


$\rho \notin$ zero polytope

$\Rightarrow E(\rho) > 0$

$$E(|z\rangle) = E\Big(\; |\phi_0\rangle + z\,|\phi_1\rangle \; \Big) = 0$$

degree 4 — rank 2



ρ is *two-root*

$$E(|z\rangle) = E\Big(\ |\phi_0\rangle + z\,|\phi_1\rangle \ \Big) = 0$$

rank 2

Entanglement as a distance

$$\underbrace{E\left(\frac{|\phi_0\rangle + \omega\,|\phi_1\rangle}{\sqrt{1+|\omega|^2}}\right)}_{|P(\omega)|}$$

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polynomial

$$N\,\frac{1}{\sqrt{1+|\omega|^2}^{\,d}}\,|(\omega - z_1)\cdots(\omega - z_d)|$$

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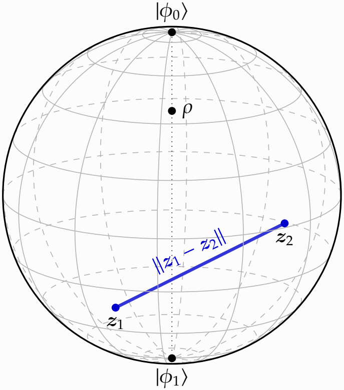
polynomial

$$N\,\frac{1}{\sqrt{1+|\omega|^2}^{\,d}}\,|(\omega - z_1)\cdots(\omega - z_d)|$$

normalisation

$$N\prod_i \sqrt{1+|z_i|^2}\,\frac{\prod_i |\omega - z_i|}{\sqrt{1+|\omega|^2}^{\,d}\,\prod_i\sqrt{1+|z_i|^2}}$$

Entanglement as a distance



chordal distance

$$\|z_1 = z_2\| = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

$$\frac{\prod_i |\omega - z_i|}{\sqrt{1+|\omega|^2}^{\,d} \prod_i \sqrt{1+|z_i|^2}}$$

Entanglement as a distance

$$\underbrace{E\left(\frac{|\phi_0\rangle + \omega\,|\phi_1\rangle}{\sqrt{1+|\omega|^2}}\right)}_{|P(\omega)|}$$

polynomial

$$N\,\frac{1}{\sqrt{1+|\omega|^2}^{\,d}}\,|(\omega - z_1)\cdots(\omega - z_d)|$$

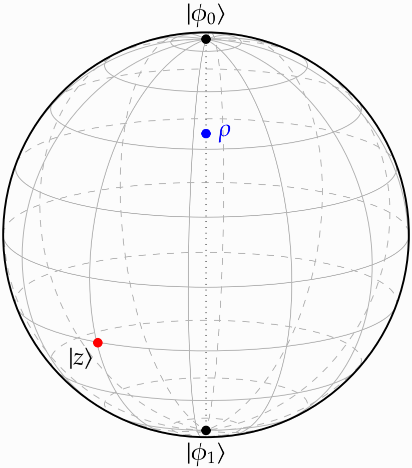
normalisation

$$N\prod_i \sqrt{1+|z_i|^2}\,\frac{\prod_i |\omega - z_i|}{\sqrt{1+|\omega|^2}^{\,d}\,\prod_i \sqrt{1+|z_i|^2}}$$

stereographic projection

$$N_\rho\,\|\boldsymbol{\omega} - \boldsymbol{z_1}\|\cdots\|\boldsymbol{\omega} - \boldsymbol{z_d}\|$$

Graphical representation of one-root states



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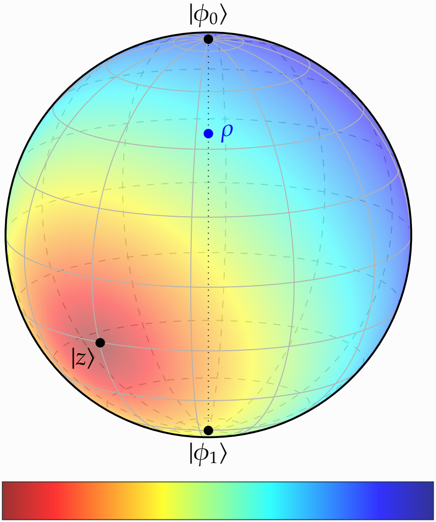

smaller distance to $|z\rangle$ (smaller entanglement) — larger distance to $|z\rangle$ (larger entanglement)

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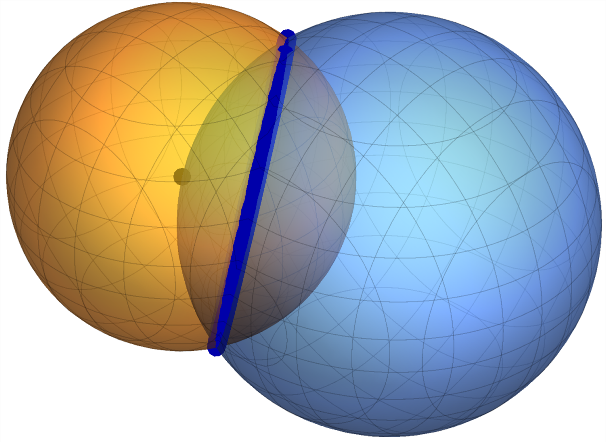
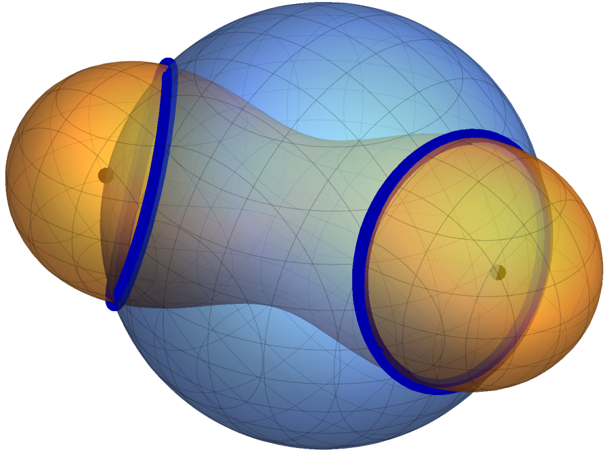


Curves of constant entanglement of one-root states

Graphical representation of two-root states



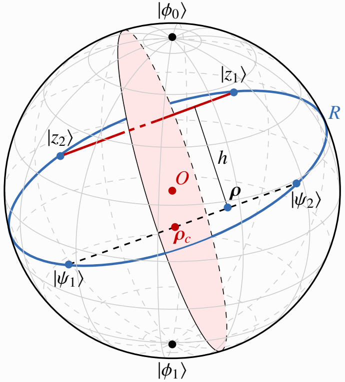
Curves of constant entanglement of two-root states

Evaluation of mixed-state entanglement for two-root states

General two-root states:

$$\rho = p_1 |\psi_1\rangle \langle\psi_1| + p_2 |\psi_2\rangle \langle\psi_2|$$

S. Hill and W. K. Wootters, PRL **78**, 5022 (1997)



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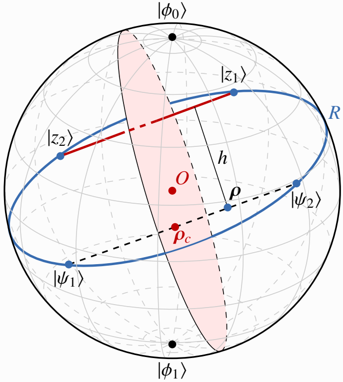
$$\begin{aligned} f(\rho) &= p_1\, C(\left|\psi_1\right\rangle) + p_2\, C(\left|\psi_2\right\rangle) \\ &= C(\left|\psi_1\right\rangle) \\ &= N_\rho \left\|\boldsymbol{\psi}_1 - \boldsymbol{z}_1\right\| \left\|\boldsymbol{\psi}_1 - \boldsymbol{z}_2\right\| \end{aligned}$$



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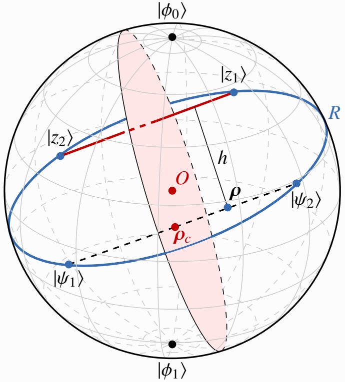

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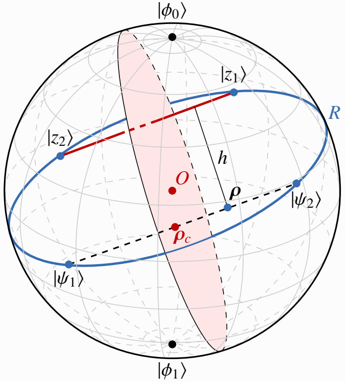

$$\begin{aligned} f(\rho) &= p_1\, C(|\psi_1\rangle) + p_2\, C(|\psi_2\rangle) \\ &= C(|\psi_1\rangle) \\ &= N_\rho\, \|\boldsymbol{\psi}_1 - \boldsymbol{z}_1\|\, \|\boldsymbol{\psi}_1 - \boldsymbol{z}_2\| \\ &= 2\, N_\rho\, R\, h \end{aligned}$$

f is the largest convex function on ρ s.t. $f(|\psi\rangle) = C(|\psi\rangle)\ \forall\, |\psi\rangle$

$$\Rightarrow\ C(\rho) = 2\, N_\rho\, R\, h$$

Evaluation of mixed-state entanglement for any measure

$$C\Big(\,|\phi_0\rangle + \omega\,|\phi_1\rangle\,\Big) = N_\rho\,\|\omega - z_1\|\,\|\omega - z_2\|$$

$$\tau\Big(\,|\phi_0\rangle + \omega\,|\phi_1\rangle\,\Big) = N_\rho\,\|\omega - z_1\|\,\|\omega - z_2\|\,\|\omega - z_3\|\,\|\omega - z_4\|$$

Evaluation of mixed-state entanglement for any measure

$$C\Big(|\phi_0\rangle + \omega\,|\phi_1\rangle \Big) = N_\rho \, \|\boldsymbol{\omega} - z_1\| \, \|\boldsymbol{\omega} - z_2\|$$

$$\tau\Big(|\phi_0\rangle + \omega\,|\phi_1\rangle \Big) = N_\rho \, \|\boldsymbol{\omega} - z_1\| \, \|\boldsymbol{\omega} - z_2\| \, \|\boldsymbol{\omega} - z_3\| \, \|\boldsymbol{\omega} - z_4\|$$

Two-root:

$$\tau\Big(|\phi_0\rangle + \omega\,|\phi_1\rangle \Big) = N_\rho \left(\|\boldsymbol{\omega} - z_1\| \, \|\boldsymbol{\omega} - z_2\| \right)^2$$

Evaluation of mixed-state entanglement for any measure

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Two-root:

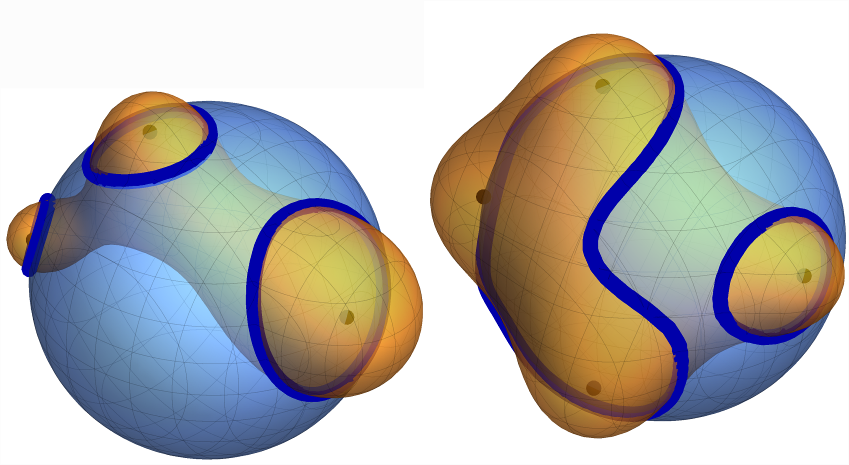
$$\tau\Big(|\phi_0\rangle + \omega\,|\phi_1\rangle \Big) = N_\rho \Big(\|\boldsymbol{\omega} - z_1\|\, \|\boldsymbol{\omega} - z_2\| \Big)^2$$

entanglement computable **exactly**

$$\tau(\rho) = N_\rho\,(2\,R\,h)^2$$

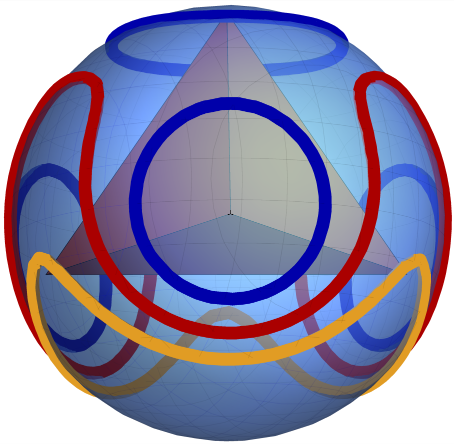
$$E_d\,(\rho) = N_\rho\,(2\,R\,h)^{d/2}$$

Geometry of entanglement with more roots



Curves of constant entanglement for three- and four-root states

Geometry of entanglement with more roots



Curves of constant three-tangle for

$$\rho = p\,|GHZ\rangle\langle GHZ| + (1-p)\,|W\rangle\langle W|$$

R. Lohmayer et al., PRL **97**, 260502 (2006)

Examples and applications

One-root example: three-tangle of 3 qubits

$$\rho = \sum_{i,j} \frac{1}{2} \left(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma} \right) |\phi_i\rangle \langle \phi_j|$$

One-root example: three-tangle of 3 qubits

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$$|\phi_0\rangle = |z\rangle = a\,|001\rangle + b\,|010\rangle + c\,|100\rangle$$

$$|\phi_1\rangle = |z'\rangle = g\,|000\rangle + t_1\,|011\rangle + t_2\,|101\rangle + t_3\,|110\rangle + e^{i\gamma} h\,|111\rangle$$

S. Tamaryan, T.-C. Wei, and D.K. Park, PRA **80**, 052315 (2009)

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one-root $\Rightarrow h = 0$, $t_3 = \dfrac{\sqrt{ct_1} + \sqrt{bt_2}}{a}$

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$$\text{one-root} \Rightarrow h = 0,\ t_3 = \frac{\sqrt{ct_1} + \sqrt{bt_2}}{a}$$

$$\sqrt{\tau}\,(\rho) = \frac{1}{2} \left| 1 - \langle \phi_0 | \rho | \phi_0 \rangle + \langle \phi_1 | \rho | \phi_1 \rangle \right| \sqrt{\tau}\,(|\phi_1\rangle)$$

$$= \sqrt{\left| \frac{g t_1 t_2}{a^9} \right| \left| \sqrt{ct_1} + \sqrt{bt_2} \right| |1 - r\cos\theta|}$$

$$\times \left| a^4 + \left[\left(\sqrt{ct_1} + \sqrt{bt_2} \right)^4 + a^2 \left(g^2 + t_1^2 + t_2^2 \right) \right]^2 \right|$$

Two-root example: three-tangle of 3 qubits

$$\rho = \sum_{i,j} \frac{1}{2} \left(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma} \right) \left| \phi_i \right\rangle \left\langle \phi_j \right|$$

$$\left| \phi_0 \right\rangle = \left| z_1 \right\rangle = a \left| 001 \right\rangle + b \left| 010 \right\rangle + c \left| 100 \right\rangle$$
$$\left| \phi_1 \right\rangle = \left| z_2 \right\rangle = d \left| 110 \right\rangle + e \left| 101 \right\rangle + f \left| 011 \right\rangle$$

E. Jung, M.-R. Hwang, D. K. Park, and J.-W. Son, PRA **79**, 024306 (2009)

Two-root example: three-tangle of 3 qubits

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$$\begin{aligned}\sqrt{\tau}(\rho) =& 2\,N_\rho\,R\,h \\ =& 2\sqrt{|a^2d^2 + b^2e^2 + c^2f^2 - 2(abde + acdf + bcef)|} \\ & \times |\langle\phi_0\,|\,\rho\,|\,\phi_1\rangle|\end{aligned}$$

SLOCC classification of four qubits

$$\rho_{ABC} = \mathrm{Tr}_D \left|\Psi_{ABCD}\right\rangle \left\langle\Psi_{ABCD}\right|$$

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Four qubits can be entangled in **nine** different ways.

F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, PRA **65**, 052112 (2002)

$$\left|\Phi_{ABCD}\right\rangle = L \left|G_\mu\right\rangle_{a,b,c,d} \qquad \text{with } \mu \in \{1,\ldots,9\},$$

$$\rho_{ABC} = \mathrm{Tr}_D \left[L \left|G_\mu\right\rangle\left\langle G_\mu\right| L^\dagger\right] \qquad L \in \mathsf{SL}(2,\mathbb{C})^{\otimes 4}$$

$$a,b,c,d \in \mathbb{C}$$

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$$a,b,c,d \in \mathbb{C}$$

$\left\|G_1\right\rangle_{a,b,c,d}$	✗	
$\left\|G_2\right\rangle_{a,b,c}$	✗	
$\left\|G_3\right\rangle_{a,b}$	✓	for $D = 2$ or 4
$\left\|G_4\right\rangle_{a,b}$	✗	
$\left\|G_5\right\rangle_{a}$	✓	for $D = 2$ or 4
$\left\|G_6\right\rangle_{a}$	✓	for $D = 2$, 3, or 4
$\left\|G_7\right\rangle$	✓	for $D = 2$, 3, or 4
$\left\|G_8\right\rangle$	✓	for $D = 2$, 3, or 4
$\left\|G_9\right\rangle$	✗	

Monogamy relations

Conventional monogamy (CKW)

V. Coffman, J. Kundu, and W.K. Wootters, PRA **61**, 052306 (2000)

$$\tau_{1|2\,3} \geq \tau_{1|2} + \tau_{1|3}$$

Generalised monogamy

T. J. Osborne and F. Verstraete, PRL **96**, 220503 (2006)

$$\tau_{1|2\cdots n} \geq \sum_{j=2}^{n} \tau_{1|j}$$

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Strong monogamy

BR, S. Di Martino, S. Lee, and G. Adesso, PRL **113**, 110501 (2014)

$$\tau_{1|2\cdots n} \overset{?}{\geq} \sum_{j=2}^{n} \tau_{1|j} + \sum_{\substack{j=2 \\ k>j}}^{n} \tau_{1|j|k} + \ldots + \sum_{j=2}^{n} \tau_{1|\cdots|j-1|j+1|\cdots|n}$$

Strong monogamy for 4 qubits

$$\tau_{1|2\,3\,4} \overset{?}{\geq} \tau_{1|2} + \tau_{1|3} + \tau_{1|4} + \tau_{1|2|3} + \tau_{1|2|4} + \tau_{1|3|4}$$

$$\tau(\rho) = \left[\min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi\rangle)} \right]^2$$

Strong monogamy for 4 qubits

$$\tau_{1|234} \not\geq \tau_{1|2} + \tau_{1|3} + \tau_{1|4} + \tau_{1|2|3} + \tau_{1|2|4} + \tau_{1|3|4}$$

only class $|G_2\rangle_{a,b,c}$ with $a=c$ or $b=c$ fails

$$\tau(\rho) = \left[\min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi\rangle)}\right]^2$$

Strong monogamy for 4 qubits

$$\tau_{1|234} \not\geq \tau_{1|2} + \tau_{1|3} + \tau_{1|4} + \tau_{1|2|3} + \tau_{1|2|4} + \tau_{1|3|4}$$

only class $|G_2\rangle_{a,b,c}$ with a=c or b=c fails

$$\tau(\rho) = \left[\min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi\rangle)}\right]^2$$

Fixes:

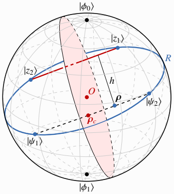
$$\tau(\rho) = \left[\min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \sqrt{\tau(|\psi\rangle)}\right]^3$$

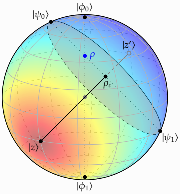
$$\tau(\rho) = \left[\min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \, \tau(|\psi\rangle)^{1/4}\right]^4$$

BR, A. Osterloh, and G. Adesso, PRA 93, 052338 (2016)

Conclusions

- Quantification of entanglement has a geometric representation, allowing us to visualise it and apply geometric methods.
- Convex roof-extended polynomial entanglement measures can be quantified **exactly** for all rank-2 states with only one or two unentangled states in their range.
- One- and two-root states occur in the marginals of several classes of four-qubit states, important in understanding the properties of entanglement monogamy.





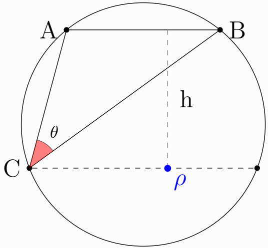
Further questions

- Where else can we find and apply one- and two-root states?
- How do the simplified conditions (one-root, two-root) generalise to states with higher rank?
- Are there classes of higher-dimensional systems that exhibit such properties?

Thank you

PRA **94**, 022324 (2016)
[arXiv:1606.06184]

Finding the formula for mixed-state concurrence



$$\begin{aligned} f(\rho) &= |\boldsymbol{CA}||\boldsymbol{CB}| \\ &= \frac{|\boldsymbol{AB}|\, h}{\sin\theta} \\ &= 2Rh \end{aligned} \tag{1}$$

SLOCC classification

$$|G^1_{abcd}\rangle = \tfrac{a+d}{2}(|0000\rangle + |1111\rangle) + \tfrac{a-d}{2}(|0011\rangle + |1100\rangle)$$
$$+\tfrac{b+c}{2}(|0101\rangle + |1010\rangle) + \tfrac{b-c}{2}(|0110\rangle + |1001\rangle)$$
$$|G^2_{abc}\rangle = \tfrac{a+b}{2}(|0000\rangle + |1111\rangle) + \tfrac{a-b}{2}(|0011\rangle + |1100\rangle)$$
$$+c(|0101\rangle + |1010\rangle) + |0110\rangle$$
$$|G^3_{ab}\rangle = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle)$$
$$+ |0110\rangle + |0011\rangle$$
$$|G^4_{ab}\rangle = a(|0000\rangle + |1111\rangle) + \tfrac{a+b}{2}(|0101\rangle + |1010\rangle)$$
$$+\tfrac{a-b}{2}(|0110\rangle + |1001\rangle)$$
$$+\tfrac{i}{\sqrt{2}}(-|0001\rangle - |0010\rangle + |0111\rangle + |1011\rangle)$$
$$|G^5_{a}\rangle = a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$
$$+i\,|0001\rangle + |0110\rangle - i\,|1011\rangle$$
$$|G^6_{a}\rangle = a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle$$
$$|G^7\rangle = |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle$$
$$|G^8\rangle = |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle$$
$$|G^9\rangle = |0000\rangle + |0111\rangle$$