Strict inequalities for quantum f-divergences and Rényi divergences

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AQIS 2016, Taipei, 01.09.2016

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    $$\exists\,\Psi: \quad \Psi(\Phi(\varrho)) = \varrho, \qquad \Psi(\Phi(\sigma)) = \sigma.$$

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- How to obtain such divergences?

Classical f-divergences

- p, q probability distributions on a finite set $\mathcal{X}$

$$D_f(p\|q) := \sum_{x\in\mathcal{X}} q(x) f\left(\frac{p(x)}{q(x)}\right)$$

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- Example: $f(x) = x\log x$: relative entropy

$$D_1(p\|q) := \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

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Rényi divergences

  - Appear naturally in error exponents.

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- All desired properties hold if f is (strictly) convex.

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- $D_f^{\mathrm{meas}}(\varrho\|\sigma) := \sup_{\mathcal{M}\text{ measurement}} D_f(\mathcal{M}(\varrho)\|\mathcal{M}(\sigma))$

[1] Hiai, Mosonyi, arXiv:1604.03089

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Maximal f-divergences

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  - Closed formula:

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    Is positivity enough?

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Moreover, if $\varrho\sigma \ne \sigma\varrho$ and $\operatorname{supp}\mu_f$ is large enough, then

$$D_f^{\text{pro}}(\varrho\|\sigma) < D_f^P(\varrho\|\sigma) < D_f^{\max}(\varrho\|\sigma)$$

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Relations of the different f-divergences

- $D_f^{\mathrm{pro}}(\varrho\|\sigma) \le D_f^{\mathrm{meas}}(\varrho\|\sigma) = \sup_{\mathrm{ran}\,\Phi\ \text{commutative}} D_f(\Phi(\varrho)\|\,\Phi(\sigma))$
- $D_f^{\max}(\varrho\|\sigma) = \mathrm{Tr}\,\sigma^{1/2} f\left(\sigma^{-1/2}\varrho\sigma^{-1/2}\right)\sigma^{1/2}$
- Petz-type f-divergences:[1,2] $D_f^P(\varrho\|\sigma) := \mathrm{Tr}\,\sigma^{1/2} f(L_\varrho R_{\sigma^{-1/2}})\sigma^{1/2}$
- Theorem:[2] For every operator convex f,

$$D_f^{\mathrm{pro}}(\varrho\|\sigma) \le D_f^{\mathrm{meas}}(\varrho\|\sigma) \le D_f^P(\varrho\|\sigma) \le D_f^{\max}(\varrho\|\sigma)$$

Moreover, if $\varrho\sigma \ne \sigma\varrho$ and $\mathrm{supp}\,\mu_f$ is large enough, then

$$D_f^{\mathrm{pro}}(\varrho\|\sigma) < D_f^P(\varrho\|\sigma) < D_f^{\max}(\varrho\|\sigma)$$

$$\left[f(x) = \alpha 1_{\{0\}} + f(0^+) + ax + bx^2 + \int_{(0,+\infty)} \left(\tfrac{x}{1+t} - \tfrac{x}{x+t}\right) d\mu_f(t)\right]$$

[1] Petz 1985; [2] Hiai, Mosonyi, Petz, Bény 2011

[2] Matsumoto 2013; Hiai, Mosonyi, arXiv:1604.03089

Sufficiency

Theorem:[1,2] Φ 2-PTP, $\operatorname{supp}\varrho \subseteq \operatorname{supp}\sigma$. T.f.a.e.:

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   $|\operatorname{supp}\mu_f| \geq |\operatorname{spec}(L_\varrho R_{\sigma^{-1}}) \cup \operatorname{spec}(L_{\Phi(\varrho)} R_{\Phi(\sigma)^{-1}})|$

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Corollary:[3] Preservation of $D_f^{\max}$ does not imply reversibility.

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Rényi divergences

- Petz-type Rényi divergences: $D_\alpha(\varrho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho^\alpha\sigma^{1-\alpha}$

[1] Mosonyi, Ogawa 2013; Hayashi, Tomamichel 2014
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- regularized measured Rényi divergence:

$$\overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) := \sup_{n\in\mathbb{N}} \frac{1}{n} D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n})$$

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D_α^*: sandwiched Rényi divergence[2]

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Rényi divergences

- Theorem:[1]

$$D_\alpha^{\mathrm{pro}}(\varrho\|\sigma) = D_\alpha^{\mathrm{meas}}(\varrho\|\sigma) \le \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) = D_\alpha^*(\varrho\|\sigma) \le D_\alpha(\varrho\|\sigma)$$

for $\alpha \ge 1/2$, and

$$D_\alpha^*(\varrho\|\sigma) \le D_\alpha^{\mathrm{pro}}(\varrho\|\sigma) = D_\alpha^{\mathrm{meas}}(\varrho\|\sigma) \le \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) \le D_\alpha(\varrho\|\sigma)$$

for $\alpha \in (0, 1/2)$

[1] Berta, Fawzi, Tomamichel 2015; Hiai, Mosonyi arXiv:1604.03089

Rényi divergences

- Theorem:[1] If $\varrho\sigma \neq \sigma\varrho$ then

$$D_\alpha^{\mathrm{pro}}(\varrho\|\sigma) = D_\alpha^{\mathrm{meas}}(\varrho\|\sigma) < \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) = D_\alpha^*(\varrho\|\sigma) < D_\alpha(\varrho\|\sigma)$$

for $\alpha > 1/2$, $\alpha \neq 1$, and

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for $\alpha \in (0, 1/2)$, and $D_\alpha^{\mathrm{meas}}(\varrho\|\sigma) < D_\alpha(\varrho\|\sigma)$.

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- $\overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) =?$ for $\alpha \in (0, 1/2)$

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- $\overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma) =?$ for $\alpha \in (0, 1/2)$

- Corollary: $D_\alpha^{\mathrm{meas}}(\varrho\|\sigma)$ is not additive for $\alpha > 1/2$

$$\varrho\sigma \neq \sigma\varrho \implies \exists n : \quad nD_\alpha^{\mathrm{meas}}(\varrho\|\sigma) < D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}).$$

[1] Berta, Fawzi, Tomamichel 2015; Hiai, Mosonyi arXiv:1604.03089

Summary

- There are various ways to define f-divergences and Rényi divergences for quantum states.

- Desirable properties are non-trivial to verify/not always known.

- Strict inequalities can be established/are expected for non-commuting states.

  Related to the sufficiency property.

- Open questions (e.g., projective measurements vs. POVM; level of positivity of maps).

Hiai, Mosonyi: arXiv:1604.03089