An Improved Semidefinite Programming Upper Bound on Distillable Entanglement and Nonadditivity of Rains' Bound

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Based on: **arXiv:1601.07940 and arXiv:1605.00348**

AQIS'2016

Partially supported by Australian Research Council (ARC) DP12 and FT12.

August 30, 2016

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- However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.
- **Entanglement of distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state ρ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

- The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used. $LOCC \subsetneq SEP \subsetneq PPT$.
- **PPT-assisted entanglement of distillation** (Rains 1999, 2001):

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An Upper bound: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

- **Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005): $E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1$.

    - (Rains, 2001; Vidal and Werner 2002): $E_D(\rho_{AB}) \le E_\Gamma(\rho_{AB}) \le E_N(\rho_{AB})$.
    - E_N has many nice properties (see later) and remains to be the best efficiently computable upper bound to E_Γ.
    - The **Negativity** $N(\rho_{AB}) = (\|\rho_{AB}^{T_B}\|_1 - 1)/2$ (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002, Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.

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A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- Primal SDP:

$$E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^{T_B}\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}. \tag{1}$$

- Dual SDP:

$$\begin{aligned} E_W(\rho_{AB}) &= \max \ \log_2 \operatorname{Tr} \rho_{AB} R_{AB}, \\ &\text{s.t.} \quad |R_{AB}^{T_B}| \leq I_{AB}, R_{AB} \geq 0. \end{aligned} \tag{2}$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  - ii) **Upper bound on PPT distillable entanglement**: $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  - iii) **Detecting genuine PPT distillable entanglement**: $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., ρ_{AB} is PPT distillable.
  - iv) **Non-increasing in average under PPT and LOCC operations**: $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if ρ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
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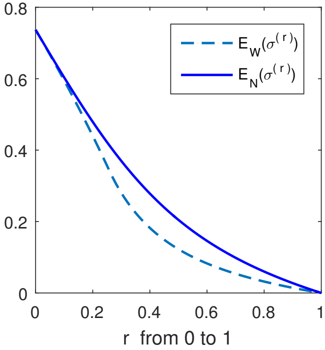
Example 1: Consider a class of two-qubit states

$$\sigma_{AB}^{(r)} = r|v_0\rangle\langle v_0| + (1-r)|v_1\rangle\langle v_1|, 0 < r < 1,$$

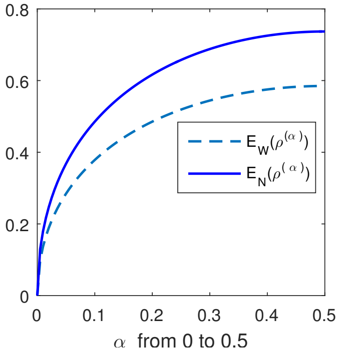
where $|v_0\rangle = 1/\sqrt{2}(|10\rangle - |11\rangle)$ and $|v_1\rangle = 1/\sqrt{3}(|00\rangle + |10\rangle + |11\rangle)$. The fact that $E_W(\sigma^{(r)}) < E_N(\sigma^{(r)})$ is shown in the following figure:



Example 2: Consider a class of $3 \otimes 3$ states $\rho_{AB}^{(\alpha)} = 1/3 \sum_{m=0}^{2} U^m |\psi_0\rangle\langle\psi_0| (U^\dagger)^m, 0 < \alpha \leq 0.5$, where $|\psi_0\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}|11\rangle$ and $U = X^\dagger \otimes X$ and $X = \sum_{i=0}^{2} |i \oplus 1\rangle\langle i|$. Then

$$E_\Gamma(\rho_{AB}^{(\alpha)}) \leq E_W(\rho_{AB}^{(\alpha)}) < E_N(\rho_{AB}^{(\alpha)}).$$

In particular, $E_\Gamma(\rho^{(0.5)}) = E_W(\rho^{(0.5)}) = \log_2 3/2 < \log_2 5/3 = E_N(\rho^{(0.5)})$.



Lower bound: PPT-assisted deterministic entanglement distillation

- **Deterministic Entanglement Distillation:** How to distill maximally entangled states exactly from a mixed state? "zero-error version of entanglement distillation".
- (Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).
- **One-copy PPT-assisted deterministic distillation rate:**

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max\{\log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \mathrm{PPT}\}.$$

  Asymptotic PPT-assisted deterministic distillation rate:

$$E_{\Gamma,0}(\rho_{AB}) := \sup_{n\geq 1} \frac{E_{\Gamma,0}^{(1)}(\rho^{\otimes n})}{n} = \lim_{n\geq 1} \frac{E_{\Gamma,0}^{(1)}(\rho^{\otimes n})}{n}.$$

- The one-copy rate can be computed by a rather simple SDP:

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max_R -\log_2 \|R_{AB}^{T_B}\|_\infty,$$

$$\text{s.t. } P_{AB} \leq R_{AB} \leq I_{AB},$$

  where P_{AB} is the projector on the support of ρ_{AB}.
- Clearly we have

$$E_{\Gamma,0}^{(1)} \leq E_{\Gamma,0} \leq E_\Gamma \leq E_W \leq E_N,$$

  and the first three inequalities become an equality while the last one is strict for $\rho_{AB}^{(0.5)}$.

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$$R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \mathrm{tr}|\sigma^{T_B}| \leq 1,$$

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Rains' bound and the relative entropy of entanglement

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Rains' bound is not additive

- **A conjecture** (Audenaert, De Moor, Vollbrecht and Werner 2002): Rains' bound is always additive;
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i) Construct a $2 \otimes 2$ state ρ so that we can explicitly find a PPT state σ such that $R(\rho) = E_R(\rho) = S(\rho||\sigma)$ using a technique in (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011).

ii) Find an upper bound $E_R^+(\rho^{\otimes 2})$ on $E_R(\rho^{\otimes 2})$ by using an algorithm developed in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015). More precisely, the algorithm produces a PPT state σ_0 so that $E_R^+(\rho^{\otimes 2}) = S(\rho^{\otimes 2}||\sigma_0)$.

iii) Achieve the goal by directly showing

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Rains' bound is not additive: Proof ideas (cont.)

Firstly, we construct two-qubit states ρ_r and σ_r such that $R(\rho_r) = S(\rho_r || \sigma_r)$ by use of a technique from (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011). Choose

$$\sigma_r = \frac{1}{4}|00\rangle\langle 00| + \frac{1}{8}|11\rangle\langle 11| + r|01\rangle\langle 01| + (\frac{5}{8} - r)|10\rangle\langle 10| + \frac{1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|).$$

The positivity of σ_r requires that $\frac{5-\sqrt{17}}{16} \leq r \leq \frac{5+\sqrt{17}}{16}$. Assume that $r \geq 5/8 - r$ and we can further choose $0.3125 \leq r \leq 0.57$ for simplicity. Then

$$\begin{aligned}\rho_r =& \frac{1}{8}|00\rangle\langle 00| + x|01\rangle\langle 01| + \frac{7-8x}{8}|10\rangle\langle 10| \\ &+ \frac{32r^2 - (6+32x)r + 10x + 1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|)\end{aligned}$$

with

$$x = r + \frac{32r^2 - 10r + 1}{256r^2 - 160r + 33} + \frac{(16r-5)y^{-1}}{32\ln(5/8 - y) - 32\ln(5/8 + y)}$$

and $y = (4r^2 - 5r/2 + 33/64)^{1/2}$. We set $0.3125 \leq r \leq 0.5480$ to ensure the positivity of ρ_r.

Let us first choose $r_0 = 0.547$, the Rains' bound of ρ_{r_0} is given by

$$R(\rho_{r_0}) = E_R(\rho_{r_0}) = S(\rho_{r_0}||\sigma_{r_0}) \simeq 0.3891999.$$

Furthermore, applying the algorithm in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015), we can find a PPT state σ_0 such that

$$E_R^+(\rho_{r_0}^{\otimes 2}) = S(\rho_{r_0}^{\otimes 2}||\sigma_0) \simeq 0.7683307.$$

(Note: in low dimensions, this algorithm provides an estimation $E_R^+(\rho)$ with an absolute error smaller than 10^{-3}, i.e. $E_R(\rho) \le E_R^+(\rho) \le E_R(\rho) + 10^{-3}$). The relative entropy here is calculated based on the Matlab function "logm" and the function "Entropy" in QETLAB (Nathaniel Johnston 2015). In this case, the accuracy is guaranteed by the fact $\|e^{\text{logm}(\sigma_{r_0})} - \sigma_{r_0}\|_1 \le 10^{-16}$ and $\|e^{\text{logm}(\sigma_0)} - \sigma_0\|_1 \le 10^{-14}$. Noting that the difference between $2R(\rho_{r_0})$ and $E_R^+(\rho_{r_0}^2)$ is already 1.00691×10^{-2}, we can safely claim that

$$R(\rho_{r_0}^{\otimes 2}) \le E_R(\rho_{r_0}^{\otimes 2}) \le E_R^+(\rho_{r_0}^{\otimes 2}) < 2R(\rho_{r_0}).$$

It is also easy to observe that

$$E_R^\infty(\rho_{r_0}) \le \frac{1}{2} E_R(\rho_{r_0}^{\otimes 2}) < R(\rho_{r_0}).$$

When $0.45 \le r \le 0.548$, we show the gap between $2R(\rho_r)$ and $E_R^+(\rho_r^{\otimes 2})$ in the following figure:

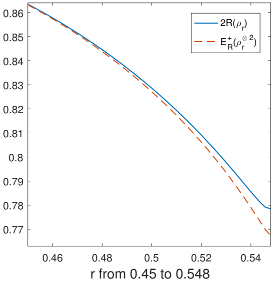

Figure : This plot demonstrates the difference between $2R(\rho_r)$ and $E_R^+(\rho_r^{\otimes 2})$ for $0.45 \le r \le 0.548$. The dashed line depicts $E_R^+(\rho_r^{\otimes 2})$ while the solid line depicts $2R(\rho_r)$.

- Regularization of Rains' bound:

$$R^{\infty}(\rho) = \inf_{k \geq 1} \frac{R(\rho^{\otimes k})}{k}.$$

- A better upper bound on distillable entanglement:

$$E_{\Gamma}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

  and the second inequality could be strict.

- **Remarks:** We were informed by one of AQIS'16 referees that Hayashi introduced the regularization of Rains' bound in his book in 2006.
- How about R^{∞} and E_R^{∞}?
- $R^{\infty}(\rho) < E_R^{\infty}(\rho)$ for any rank-2 mixed state ρ supporting on the anti-symmetric $3 \otimes 3$ space, thus the irreversibility of PPT manipulation of entanglement, the 20^{th} open problem in Quantum Information Theory on Werner's website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).

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