An Improved Semidefinite Programming Upper Bound on Distillable Entanglement and Nonadditivity of Rains’ Bound

Xin Wang and Runyao Duan

Centre for Quantum Computation & Intelligent Systems, University of Technology Sydney (UTS), Sydney


AQIS’2016

Partially supported by Australian Research Council (ARC) DP12 and FT12.

August 30, 2016
Maximally entangled states $\Phi(k) = 1/k \sum_{i,j=1}^{k} |ii\rangle\langle jj|$ are useful physical resources in quantum teleportation, superdense coding, etc.

However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.

Entanglement of distillation (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state $\rho$ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup\{r : \lim_{n \to \infty} \inf_{\Lambda \in \text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^r n)\|_1 = 0\}.$$ 

The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used. $\text{LOCC} \subsetneq \text{SEP} \subsetneq \text{PPT}.$

PPT-assisted entanglement of distillation (Rains 1999, 2001):

$$E_\Gamma(\rho_{AB}) = \sup\{r : \lim_{n \to \infty} \inf_{\Lambda \in \text{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^r n)\|_1 = 0\}.$$
Overview of entanglement distillation

- Maximally entangled states $\Phi(k) = 1/k \sum_{i,j=1}^{k} |ii\rangle\langle jj|$ are useful physical resources in quantum teleportation, superdense coding, etc.

- However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.

- Entanglement of distillation (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state $\rho$ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup\{r : \lim_{n \to \infty} \inf_{\Lambda \in \text{LOCC}} \|\Lambda(\rho_{AB}^\otimes n) - \Phi(2^rn)\|_1 = 0\}.$$ 

- The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used. $\text{LOCC} \subsetneq \text{SEP} \subsetneq \text{PPT}$.

- PPT-assisted entanglement of distillation (Rains 1999, 2001):

$$E_G(\rho_{AB}) = \sup\{r : \lim_{n \to \infty} \inf_{\Lambda \in \text{PPT}} \|\Lambda(\rho_{AB}^\otimes n) - \Phi(2^rn)\|_1 = 0\}.$$
Overview of entanglement distillation

- Maximally entangled states $\Phi(k) = \frac{1}{k} \sum_{i,j=1}^{k} |ii\rangle\langle jj|$ are useful physical resources in quantum teleportation, superdense coding, etc.

- However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.

- **Entanglement of distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state $\rho$ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{LOCC}} \| \Lambda(\rho_{AB}^\otimes n) - \Phi(2^r n) \|_1 = 0 \}.$$

- The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used.

$$LOCC \subsetneq SEP \subsetneq PPT.$$

- **PPT-assisted entanglement of distillation** (Rains 1999, 2001):

$$E_\Gamma(\rho_{AB}) = \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{PPT}} \| \Lambda(\rho_{AB}^\otimes n) - \Phi(2^r n) \|_1 = 0 \}.$$
Overview of entanglement distillation

- Maximally entangled states $\Phi(k) = 1/k \sum_{i,j=1}^{k} |ii\rangle\langle jj|$ are useful physical resources in quantum teleportation, superdense coding, etc.

- However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.

- **Entanglement of distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state $\rho$ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in LOCC} \| \Lambda(\rho_{AB}^\otimes n) - \Phi(2^r n) \|_1 = 0 \}.$$

- The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used. $LOCC \subsetneq SEP \subsetneq PPT$.

- **PPT-assisted entanglement of distillation** (Rains 1999, 2001):

$$E_\Gamma(\rho_{AB}) = \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in PPT} \| \Lambda(\rho_{AB}^\otimes n) - \Phi(2^r n) \|_1 = 0 \}.$$
Overview of entanglement distillation

- Maximally entangled states $\Phi(k) = 1/k \sum_{i,j=1}^{k} |ii\rangle\langle jj|$ are useful physical resources in quantum teleportation, superdense coding, etc.

- However, in practice, only partially entangled pure or noisy mixed states are available. We need extract maximally entangled states from them.

- **Entanglement of distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, PRA, 1999): the highest rate at which one can obtain $\Phi(2)$ (EPR pairs, or ebits) from the given state $\rho$ by local operations and classical communications (LOCC),

$$E_D(\rho_{AB}) = \sup\{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{LOCC}} \|\Lambda(\rho_{AB}^\otimes n) - \Phi(2^rn)\|_1 = 0 \}.$$ 

- The structure of LOCC is very complicated, and usually more tractable operations such as separable operations (SEP) or operations completely preserving the positivity of partial transpose (PPT) are used.

  $$\text{LOCC} \subsetneq \text{SEP} \subsetneq \text{PPT}.$$ 

- **PPT-assisted entanglement of distillation** (Rains 1999, 2001):

$$E_\Gamma(\rho_{AB}) = \sup\{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{PPT}} \|\Lambda(\rho_{AB}^\otimes n) - \Phi(2^rn)\|_1 = 0 \}.$$
How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

**Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005): 
\[ E_N(\rho_{AB}) = \log_2 \|\rho^T_B\|_1. \]

- (Rains, 2001; Vidal and Werner 2002): \( E_D(\rho_{AB}) \leq E_\Gamma(\rho_{AB}) \leq E_N(\rho_{AB}) \).
- \( E_N \) has many nice properties (see later) and remains to be the best efficiently computable upper bound to \( E_\Gamma \).
- The **Negativity** \( N(\rho_{AB}) = (\|\rho^T_B\|_1 - 1)/2 \) (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002; Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.
How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

**Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):
\[ E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^T\|_1. \]

(Rains, 2001; Vidal and Werner 2002): \[ E_D(\rho_{AB}) \leq E_\Gamma(\rho_{AB}) \leq E_N(\rho_{AB}). \]

\( E_N \) has many nice properties (see later) and remains to be the best efficiently computable upper bound to \( E_\Gamma \).

The **Negativity** \( N(\rho_{AB}) = (\|\rho_{AB}^T\|_1 - 1)/2 \) (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002, Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.
How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

**Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):
\[ E_N(\rho_{AB}) = \log_2 \left\| \rho_{T_B}^{AB} \right\|_1. \]

(Rains, 2001; Vidal and Werner 2002): \[ E_D(\rho_{AB}) \leq E_\Gamma(\rho_{AB}) \leq E_N(\rho_{AB}). \]

\( E_N \) has many nice properties (see later) and remains to be the best efficiently computable upper bound to \( E_\Gamma \).

The **Negativity** \( N(\rho_{AB}) = (\|\rho_{T_B}^{AB}\|_1 - 1)/2 \) (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002, Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.
How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

**Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):

\[ E_N(\rho_{AB}) = \log_2 \|\rho_{T_B}^{AB}\|_1. \]

(Rains, 2001; Vidal and Werner 2002): \( E_D(\rho_{AB}) \leq E_\Gamma(\rho_{AB}) \leq E_N(\rho_{AB}). \)

\( E_N \) has many nice properties (see later) and remains to be the best efficiently computable upper bound to \( E_\Gamma \).

The **Negativity** \( N(\rho_{AB}) = (\|\rho_{T_B}^{AB}\|_1 - 1)/2 \) (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002, Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.
How to evaluate the distillable entanglement (by any of LOCC, SEP, or PPT) are formidable. Only known for very limited cases.

**Logarithmic negativity** (Vidal and Werner 2002; Plenio 2005):

\[ E_N(\rho_{AB}) = \log_2 \| \rho_{AB}^T \|_1. \]

(Rains, 2001; Vidal and Werner 2002): \( E_D(\rho_{AB}) \leq E_\Gamma(\rho_{AB}) \leq E_N(\rho_{AB}) \).

\( E_N \) has many nice properties (see later) and remains to be the best efficiently computable upper bound to \( E_\Gamma \).

The **Negativity** \( N(\rho_{AB}) = (\| \rho_{AB}^T \|_1 - 1)/2 \) (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998) is an entanglement monotone (Vidal and Werner 2002, Eisert 2006, Plenio 2005), but is not directly related to entanglement of distillation.
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**
  $$E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \text{ s.t. } X_{AB} \geq \rho_{AB}. \quad (1)$$

- **Dual SDP:**
  $$E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB},$$
  $$\text{s.t. } |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0. \quad (2)$$

- **Properties of $E_W$:**
  i) Additivity: $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  ii) Upper bound on PPT distillable entanglement: $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  iii) Detecting genuine PPT distillable entanglement: $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) Non-increasing in average under PPT and LOCC operations:
  $$E_W(\rho) \geq \sum_i p_i E_W(\rho_i) \text{ if } \rho \text{ can be transformed to } \{(p_i, \rho_i)\} \text{ via LOCC (or PPT).}$$
  v) Improved over logarithmic negativity: $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**

  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}. \tag{1}
  \]

- **Dual SDP:**

  \[
  E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB}, \tag{2}
  \]

  \[
  \text{s.t.} \quad |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0.
  \]

- **Properties of $E_W$:**
  
  i) Additivity: $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  
  ii) Upper bound on PPT distillable entanglement: $E_T(\rho_{AB}) \leq E_W(\rho_{AB})$.
  
  iii) Detecting genuine PPT distillable entanglement: $E_W(\rho_{AB}) > 0$ iff $E_T(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  
  iv) Non-increasing in average under PPT and LOCC operations: $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
  
  v) Improved over logarithmic negativity: $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**
  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^{TB}\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}.
  \] (1)

- **Dual SDP:**
  \[
  E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB},
  \]
  \[
  \text{s.t.} \quad |R_{AB}^{TB}| \leq I_{AB}, R_{AB} \geq 0.
  \] (2)

- **Properties of $E_W$:**
  i) **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  ii) **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  iii) **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) **Non-increasing in average under PPT and LOCC operations:** $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
  v) **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**
  
  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}.
  \]

- **Dual SDP:**
  
  \[
  E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB},
  \]
  \[
  \text{s.t.} \quad |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0.
  \]

- **Properties of $E_W$:**
  
  i) **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'}).$
  
  ii) **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB}).$
  
  iii) **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  
  iv) **Non-increasing in average under PPT and LOCC operations:** $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to \{(\rho_i, \rho_i)\} via LOCC (or PPT).
  
  v) **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**
  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^{TB}\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}. \tag{1}
  \]

- **Dual SDP:**
  \[
  E_W(\rho_{AB}) = \max \log_2 \Tr \rho_{AB} R_{AB},
  \]
  \[
  \text{s.t.} \quad |R_{AB}^{TB}| \leq I_{AB}, R_{AB} \geq 0. \tag{2}
  \]

- **Properties of $E_W$:**
  i) **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  ii) **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  iii) **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) **Non-increasing in average under PPT and LOCC operations:**
      $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
  v) **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
**A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$**

- **Primal SDP:**
  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}.
  \]  
  (1)

- **Dual SDP:**
  \[
  E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB},
  \]
  \[
  \text{s.t.} \quad |R_{AB}^{TB}| \leq I_{AB}, R_{AB} \geq 0.
  \]  
  (2)

- **Properties of $E_W$:**
  1. **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  2. **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  3. **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  4. **Non-increasing in average under PPT and LOCC operations:** $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
  5. **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**
  \[
  E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}. \tag{1}
  \]

- **Dual SDP:**
  \[
  E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB}, \quad \text{s.t.} \quad |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0. \tag{2}
  \]

- **Properties of $E_W$:**
  i) **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  ii) **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  iii) **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) **Non-increasing in average under PPT and LOCC operations:**
  \[
  E_W(\rho) \geq \sum_i p_i E_W(\rho_i) \text{ if } \rho \text{ can be transformed to } \{(p_i, \rho_i)\} \text{ via LOCC (or PPT).}
  \]
  v) **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- **Primal SDP:**

  $$E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \quad \text{s.t.} \quad X_{AB} \geq \rho_{AB}. \quad (1)$$

- **Dual SDP:**

  $$E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB},$$
  $$\text{s.t.} \quad |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0. \quad (2)$$

- **Properties of $E_W$:**
  i) **Additivity:** $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'})$.
  ii) **Upper bound on PPT distillable entanglement:** $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB})$.
  iii) **Detecting genuine PPT distillable entanglement:** $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) **Non-increasing in average under PPT and LOCC operations:** $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to $\{(p_i, \rho_i)\}$ via LOCC (or PPT).
  v) **Improved over logarithmic negativity:** $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
A better semidefinite programming (SDP) upper bound: $E_W(\rho_{AB})$

- Primal SDP:
  \[ E_W(\rho_{AB}) = \min \log_2 \|X_{AB}^T\|_1, \text{ s.t. } X_{AB} \geq \rho_{AB}. \]  
  (1)

- Dual SDP:
  \[ E_W(\rho_{AB}) = \max \log_2 \text{Tr} \rho_{AB} R_{AB}, \]  
  \[ \text{s.t. } |R_{AB}^T| \leq I_{AB}, R_{AB} \geq 0. \]  
  (2)

- Properties of $E_W$:
  i) **Additivity**: $E_W(\rho_{AB} \otimes \sigma_{A'B'}) = E_W(\rho_{AB}) + E_W(\sigma_{A'B'}).$
  ii) **Upper bound on PPT distillable entanglement**: $E_\Gamma(\rho_{AB}) \leq E_W(\rho_{AB}).$
  iii) **Detecting genuine PPT distillable entanglement**: $E_W(\rho_{AB}) > 0$ iff $E_\Gamma(\rho_{AB}) > 0$, i.e., $\rho_{AB}$ is PPT distillable.
  iv) **Non-increasing in average under PPT and LOCC operations**: $E_W(\rho) \geq \sum_i p_i E_W(\rho_i)$ if $\rho$ can be transformed to \{(\rho_i, \rho_i)\} via LOCC (or PPT).
  v) **Improved over logarithmic negativity**: $E_W(\rho_{AB}) \leq E_N(\rho_{AB})$, and the inequality is strict in general.

- $E_N$ has all above properties except v)!!!
Example 1: Consider a class of two-qubit states

\[
\sigma_{AB}^{(r)} = r|v_0\rangle \langle v_0| + (1 - r)|v_1\rangle \langle v_1|, \quad 0 < r < 1,
\]

where \(|v_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)\) and \(|v_1\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |10\rangle + |11\rangle)\). The fact that \(E_W(\sigma^{(r)}) < E_N(\sigma^{(r)})\) is shown in the following figure:
**Example 2:** Consider a class of $3 \otimes 3$ states 

$$\rho_{AB}^{(\alpha)} = \frac{1}{3} \sum_{m=0}^{2} U^m |\psi_0\rangle\langle\psi_0| (U^\dagger)^m, \quad 0 < \alpha \leq 0.5,$$

where 

$$|\psi_0\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}|11\rangle$$

and 

$$U = X^\dagger \otimes X$$

and 

$$X = \sum_{i=0}^{2} |i \oplus 1\rangle \langle i|.$$ 

Then 

$$E_\Gamma(\rho_{AB}^{(\alpha)}) \leq E_W(\rho_{AB}^{(\alpha)}) < E_N(\rho_{AB}^{(\alpha)}).$$

In particular, 

$$E_\Gamma(\rho^{(0.5)}) = E_W(\rho^{(0.5)}) = \log_2 3/2 < \log_2 5/3 = E_N(\rho^{(0.5)}).$$
Deterministic Entanglement Distillation: How to distill maximally entangled states exactly from a mixed state? “zero-error version of entanglement distillation”.

(Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).

One-copy PPT-assisted deterministic distillation rate:
\[
E^{(1)}_{\Gamma,0}(\rho_{AB}) = \max\{\log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \text{PPT}\}.
\]

Asymptotic PPT-assisted deterministic distillation rate:
\[
E_{\Gamma,0}(\rho_{AB}) := \sup_{n \geq 1} \frac{E^{(1)}_{\Gamma,0}(\rho_{AB}^\otimes n)}{n} = \lim_{n \to \infty} \frac{E^{(1)}_{\Gamma,0}(\rho_{AB}^\otimes n)}{n}.
\]

The one-copy rate can be computed by a rather simple SDP:
\[
E^{(1)}_{\Gamma,0}(\rho_{AB}) = \max_R -\log_2 \|R_{AB}^T\|_{\infty},
\]
\[
\text{s.t. } P_{AB} \leq R_{AB} \leq I_{AB},
\]
where \(P_{AB}\) is the projector on the support of \(\rho_{AB}\).

Clearly we have
\[
E^{(1)}_{\Gamma,0} \leq E_{\Gamma,0} \leq E_{\Gamma} \leq E_W \leq E_{N},
\]
and the first three inequalities become an equality while the last one is strict for \(\rho_{AB}^{(0.5)}\).
**Deterministic Entanglement Distillation:** How to distill maximally entangled states exactly from a mixed state? “zero-error version of entanglement distillation”.

(Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).

One-copy PPT-assisted deterministic distillation rate:

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max \{ \log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \text{PPT} \}.$$  

Asymptotic PPT-assisted deterministic distillation rate:

$$E_{\Gamma,0}(\rho_{AB}) := \sup_{n \geq 1} \frac{E_{\Gamma,0}^{(1)}(\rho_{AB}^\otimes n)}{n} = \lim_{n \to \infty} \frac{E_{\Gamma,0}^{(1)}(\rho_{AB}^\otimes n)}{n}.$$  

The one-copy rate can be computed by a rather simple SDP:

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max_R - \log_2 \| R_{AB}^T \|_\infty,$$

s.t. $P_{AB} \leq R_{AB} \leq I_{AB},$

where $P_{AB}$ is the projector on the support of $\rho_{AB}$.

Clearly we have

$$E_{\Gamma,0}^{(1)} \leq E_{\Gamma,0} \leq E_{\Gamma} \leq E_W \leq E_N,$$

and the first three inequalities become an equality while the last one is strict for $\rho_{AB}^{(0.5)}$.  

---

Xin Wang and Runyao Duan, AQIS’2016
**Deterministic Entanglement Distillation:** How to distill maximally entangled states exactly from a mixed state? “zero-error version of entanglement distillation”.

(Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).

**One-copy PPT-assisted deterministic distillation rate:**
\[ E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max \{ \log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \text{PPT} \} . \]

**Asymptotic PPT-assisted deterministic distillation rate:**
\[ E_{\Gamma,0}(\rho_{AB}) := \sup_{n \geq 1} \frac{E_{\Gamma,0}^{(1)}(\rho_{AB}^\otimes n)}{n} = \lim_{n \to \infty} \frac{E_{\Gamma,0}^{(1)}(\rho_{AB}^\otimes n)}{n} . \]

The one-copy rate can be computed by a rather simple SDP:
\[ E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max_R - \log_2 \| R_{AB}^T \|_\infty , \]
\[ \text{s.t. } P_{AB} \leq R_{AB} \leq I_{AB}, \]
where \( P_{AB} \) is the projector on the support of \( \rho_{AB} \).

Clearly we have
\[ E_{\Gamma,0}^{(1)} \leq E_{\Gamma,0} \leq E_{\Gamma} \leq E_W \leq E_N , \]
and the first three inequalities become an equality while the last one is strict for \( \rho_{AB}^{(0.5)} \)
Deterministic Entanglement Distillation: How to distill maximally entangled states exactly from a mixed state? “zero-error version of entanglement distillation”.

(Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).

One-copy PPT-assisted deterministic distillation rate:

\[ E^{(1)}_{\Gamma,0}(\rho_{AB}) = \max\{\log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \text{PPT}\} \]

Asymptotic PPT-assisted deterministic distillation rate:

\[ E_{\Gamma,0}(\rho_{AB}) := \sup_{n \geq 1} \frac{E^{(1)}_{\Gamma,0}(\rho_{AB}^\otimes n)}{n} = \lim_{n \to \infty} \frac{E^{(1)}_{\Gamma,0}(\rho_{AB}^\otimes n)}{n}. \]

The one-copy rate can be computed by a rather simple SDP:

\[ E^{(1)}_{\Gamma,0}(\rho_{AB}) = \max_{R} - \log_2 \| R^{TB}_{AB} \|_{\infty}, \]

s.t. \( P_{AB} \leq R_{AB} \leq I_{AB} \),

where \( P_{AB} \) is the projector on the support of \( \rho_{AB} \).

Clearly we have

\[ E^{(1)}_{\Gamma,0} \leq E_{\Gamma,0} \leq E_{\Gamma} \leq E_W \leq E_N, \]

and the first three inequalities become an equality while the last one is strict for \( \rho_{AB}^{(0.5)} \).
Deterministic Entanglement Distillation: How to distill maximally entangled states exactly from a mixed state? “zero-error version of entanglement distillation”.

(Duan, Feng, Ji, Ying, 2004; Matthews and Winter, 2008): Bipartite pure state cases (LOCC and PPT no difference!).

One-copy PPT-assisted deterministic distillation rate:

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max \{ \log_2 k : \Lambda(\rho_{AB}) = \Phi(k), \exists \Lambda \in \text{PPT} \}.$$ 

Asymptotic PPT-assisted deterministic distillation rate:

$$E_{\Gamma,0}(\rho_{AB}) := \sup_{n \geq 1} \frac{E_{\Gamma,0}^{(1)}(\rho \otimes^n)}{n} = \lim_{n \to \infty} \frac{E_{\Gamma,0}^{(1)}(\rho \otimes^n)}{n}.$$ 

The one-copy rate can be computed by a rather simple SDP:

$$E_{\Gamma,0}^{(1)}(\rho_{AB}) = \max_R -\log_2 \| R_{AB}^T \|_\infty,$$

s.t. $P_{AB} \leq R_{AB} \leq I_{AB},$ 

where $P_{AB}$ is the projector on the support of $\rho_{AB}.$

Clearly we have

$$E_{\Gamma,0}^{(1)} \leq E_{\Gamma,0} \leq E_{\Gamma} \leq E_W \leq E_N,$$

and the first three inequalities become an equality while the last one is strict for $\rho_{AB}^{(0.5)}.$
**Rains’ bound** (Rains 2001; Audenaert, De Moor, Vollbrecht and Werner 2002):

\[ R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \text{tr}|\sigma^B| \leq 1, \]

where \( S(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma) \) denotes the relative Von Neumann entropy.

(Rains 2001) Rains’ bound is the best known upper bound on the PPT distillable entanglement, i.e., \( E_\Gamma(\rho) \leq R(\rho) \).

**Relative entropy of entanglement** (Vedral, Plenio, Rippin and Knight, 1997; Vedral and Plenio 1998; Vedral, Plenio, Jacobs and Knight 1997) with respect to the PPT states:

\[ E_R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma, \sigma^T \geq 0, \text{tr}\sigma = 1. \]

The asymptotic relative entropy of entanglement is given by

\[ E_R^\infty(\rho) = \inf_{n \geq 1} \frac{1}{n} E_R(\rho^\otimes n). \]

Clearly, \( E_R(\rho) \geq R(\rho) \), and \( E_R(\rho) \) equals to \( R(\rho) \) for every two-qubit state (Miranowicz and Ishizaka 2008) or the bipartite state with one qubit subsystem (Girard, Gour and Friedland 2014).
Rains’ bound (Rains 2001; Audenaert, De Moor, Vollbrecht and Werner 2002):

\[ R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \text{tr}|\sigma^T| \leq 1, \]

where \( S(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma) \) denotes the relative von Neumann entropy.

(Rains 2001) Rains’ bound is the best known upper bound on the PPT distillable entanglement, i.e., \( E_{\Gamma}(\rho) \leq R(\rho) \).

Relative entropy of entanglement (Vedral, Plenio, Rippin and Knight, 1997; Vedral and Plenio 1998; Vedral, Plenio, Jacobs and Knight 1997) with respect to the PPT states:

\[ E_R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma, \sigma^T \geq 0, \text{tr} \sigma = 1. \]

The asymptotic relative entropy of entanglement is given by

\[ E_R^\infty(\rho) = \inf_{n \geq 1} \frac{1}{n} E_R(\rho^\otimes n). \]

Clearly, \( E_R(\rho) \geq R(\rho) \), and \( E_R(\rho) \) equals to \( R(\rho) \) for every two-qubit state (Miranowicz and Ishizaka 2008) or the bipartite state with one qubit subsystem (Girard, Gour and Friedland 2014).
Rains’ bound (Rains 2001; Audenaert, De Moor, Vollbrecht and Werner 2002):

\[ R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \text{tr}|\sigma^T| \leq 1, \]

where \( S(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma) \) denotes the relative Von Neumann entropy.

(Rains 2001) Rains’ bound is the best known upper bound on the PPT distillable entanglement, i.e., \( E_\Gamma(\rho) \leq R(\rho) \).

Relative entropy of entanglement (Vedral, Plenio, Rippin and Knight, 1997; Vedral and Plenio 1998; Vedral, Plenio, Jacobs and Knight 1997) with respect to the PPT states:

\[ E_R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma, \sigma^T \geq 0, \text{tr}\sigma = 1. \]

The asymptotic relative entropy of entanglement is given by

\[ E^\infty_R(\rho) = \inf_{n \geq 1} \frac{1}{n} E_R(\rho^\otimes n). \]

Clearly, \( E_R(\rho) \geq R(\rho) \), and \( E_R(\rho) \) equals to \( R(\rho) \) for every two-qubit state (Miranowicz and Ishizaka 2008) or the bipartite state with one qubit subsystem (Girard, Gour and Friedland 2014).
Rains’ bound (Rains 2001; Audenaert, De Moor, Vollbrecht and Werner 2002):

\[ R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \text{tr}|\sigma^{TB}| \leq 1, \]

where \( S(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma) \) denotes the relative Von Neumann entropy.

(Rains 2001) Rains’ bound is the best known upper bound on the PPT distillable entanglement, i.e., \( E_{\Gamma}(\rho) \leq R(\rho) \).

Relative entropy of entanglement (Vedral, Plenio, Rippin and Knight, 1997; Vedral and Plenio 1998; Vedral, Plenio, Jacobs and Knight 1997) with respect to the PPT states:

\[ E_R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma, \sigma^{TB} \geq 0, \text{tr}\sigma = 1. \]

The asymptotic relative entropy of entanglement is given by

\[ E_R^\infty(\rho) = \inf_{n \geq 1} \frac{1}{n} E_R(\rho^\otimes n). \]

Clearly, \( E_R(\rho) \geq R(\rho) \), and \( E_R(\rho) \) equals to \( R(\rho) \) for every two-qubit state (Miranowicz and Ishizaka 2008) or the bipartite state with one qubit subsystem (Girard, Gour and Friedland 2014).
A conjecture (Audenaert, De Moor, Vollbrecht and Werner 2002): Rains’ bound is always additive;

An open problem (Plenio and Virmani 2007): Whether Rains’ bound is always equal to the asymptotic relative entropy of entanglement?

Theorem: There exists a two-qubit state $\rho$ such that

$$R(\rho^{\otimes 2}) < 2R(\rho).$$

Meanwhile,

$$E^\infty_R(\rho) < R(\rho).$$
A conjecture (Audenaert, De Moor, Vollbrecht and Werner 2002): Rains’ bound is always additive;

An open problem (Plenio and Virmani 2007): Whether Rains’ bound is always equal to the asymptotic relative entropy of entanglement?

Theorem: There exists a two-qubit state $\rho$ such that

$$R(\rho^{\otimes 2}) < 2R(\rho).$$

Meanwhile,

$$E_\infty^R(\rho) < R(\rho).$$
A conjecture (Audenaert, De Moor, Vollbrecht and Werner 2002): Rains’ bound is always additive;

An open problem (Plenio and Virmani 2007): Whether Rains’ bound is always equal to the asymptotic relative entropy of entanglement?

Theorem: There exists a two-qubit state $\rho$ such that

$$R(\rho^\otimes 2) < 2R(\rho).$$

Meanwhile,

$$E^\infty_R(\rho) < R(\rho).$$
i) Construct a $2 \otimes 2$ state $\rho$ so that we can explicitly find a PPT state $\sigma$ such that $R(\rho) = E_R(\rho) = S(\rho||\sigma)$ using a technique in (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011).

ii) Find an upper bound $E_R^+(\rho^{\otimes 2})$ on $E_R(\rho^{\otimes 2})$ by using an algorithm developed in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015). More precisely, the algorithm produces a PPT state $\sigma_0$ so that $E_R^+(\rho^{\otimes 2}) = S(\rho^{\otimes 2}||\sigma_0)$.

iii) Achieve the goal by directly showing

$$R(\rho^{\otimes 2}) \leq E_R(\rho^{\otimes 2}) \leq E_R^+(\rho^{\otimes 2}) \leq 2E_R(\rho) = 2R(\rho).$$

iv) An example of semi-analytical and semi-numerical proof.
i) Construct a $2 \otimes 2$ state $\rho$ so that we can explicitly find a PPT state $\sigma$ such that $R(\rho) = E_R(\rho) = S(\rho \| \sigma)$ using a technique in (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011).

ii) Find an upper bound $E_R^+(\rho \otimes 2)$ on $E_R(\rho \otimes 2)$ by using an algorithm developed in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015). More precisely, the algorithm produces a PPT state $\sigma_0$ so that $E_R^+(\rho \otimes 2) = S(\rho \otimes 2 \| \sigma_0)$.

iii) Achieve the goal by directly showing

$$R(\rho \otimes 2) \leq E_R(\rho \otimes 2) \leq E_R^+(\rho \otimes 2) < 2E_R(\rho) = 2R(\rho).$$

iv) An example of semi-analytical and semi-numerical proof.
i) Construct a $2 \otimes 2$ state $\rho$ so that we can explicitly find a PPT state $\sigma$ such that $R(\rho) = E_R(\rho) = S(\rho||\sigma)$ using a technique in (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011).

ii) Find an upper bound $E_R^+(\rho^{\otimes 2})$ on $E_R(\rho^{\otimes 2})$ by using an algorithm developed in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015). More precisely, the algorithm produces a PPT state $\sigma_0$ so that $E_R^+(\rho^{\otimes 2}) = S(\rho^{\otimes 2}||\sigma_0)$.

iii) Achieve the goal by directly showing

$$R(\rho^{\otimes 2}) \leq E_R(\rho^{\otimes 2}) \leq E_R^+(\rho^{\otimes 2}) < 2E_R(\rho) = 2R(\rho).$$

iv) An example of semi-analytical and semi-numerical proof.
Rains’ bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state $\rho$ so that we can explicitly find a PPT state $\sigma$ such that $R(\rho) = E_R(\rho) = S(\rho||\sigma)$ using a technique in (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011).

ii) Find an upper bound $E_R^+(\rho^{\otimes 2})$ on $E_R(\rho^{\otimes 2})$ by using an algorithm developed in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015). More precisely, the algorithm produces a PPT state $\sigma_0$ so that $E_R^+(\rho^{\otimes 2}) = S(\rho^{\otimes 2}||\sigma_0)$.

iii) Achieve the goal by directly showing

$$R(\rho^{\otimes 2}) \leq E_R(\rho^{\otimes 2}) \leq E_R^+(\rho^{\otimes 2}) < 2E_R(\rho) = 2R(\rho).$$

iv) An example of semi-analytical and semi-numerical proof.
Firstly, we construct two-qubit states $\rho_r$ and $\sigma_r$ such that $R(\rho_r) = S(\rho_r||\sigma_r)$ by use of a technique from (Miranowicz and Ishizaka 2008; see also Gour and Friedland 2011). Choose

$$\sigma_r = \frac{1}{4} |00\rangle\langle 00| + \frac{1}{8} |11\rangle\langle 11| + r |01\rangle\langle 01| + \left( \frac{5}{8} - r \right) |10\rangle\langle 10| + \frac{1}{4\sqrt{2}} (|01\rangle\langle 10| + |10\rangle\langle 01|).$$

The positivity of $\sigma_r$ requires that $\frac{5 - \sqrt{17}}{16} \leq r \leq \frac{5 + \sqrt{17}}{16}$. Assume that $r \geq 5/8 - r$ and we can further choose $0.3125 \leq r \leq 0.57$ for simplicity. Then

$$\rho_r = \frac{1}{8} |00\rangle\langle 00| + x |01\rangle\langle 01| + \frac{7 - 8x}{8} |10\rangle\langle 10|$$

$$+ \frac{32r^2 - (6 + 32x)r + 10x + 1}{4\sqrt{2}} (|01\rangle\langle 10| + |10\rangle\langle 01|)$$

with

$$x = r + \frac{32r^2 - 10r + 1}{256r^2 - 160r + 33} + \frac{(16r - 5)y^{-1}}{32 \ln (5/8 - y) - 32 \ln (5/8 + y)}$$

and $y = (4r^2 - 5r/2 + 33/64)^{1/2}$. We set $0.3125 \leq r \leq 0.5480$ to ensure the positivity of $\rho_r$. 

Xin Wang and Runyao Duan, AQIS'2016
Let us first choose $r_0 = 0.547$, the Rains’ bound of $\rho_{r_0}$ is given by

$$R(\rho_{r_0}) = E_R(\rho_{r_0}) = S(\rho_{r_0} || \sigma_{r_0}) \simeq 0.3891999.$$ 

Furthermore, applying the algorithm in (Zinchenko, Friedland, and Gour 2010; Girard, Zinchenko, Friedland and Gour 2015), we can find a PPT state $\sigma_0$ such that

$$E^+_R(\rho_{r_0} \otimes 2) = S(\rho_{r_0} \otimes 2 || \sigma_0) \simeq 0.7683307.$$ 

(Note: in low dimensions, this algorithm provides an estimation $E^+_R(\rho)$ with an absolute error smaller than $10^{-3}$, i.e. $E_R(\rho) \leq E^+_R(\rho) \leq E_R(\rho) + 10^{-3}$).

The relative entropy here is calculated based on the Matlab function “logm” and the function “Entropy” in QETLAB (Nathaniel Johnston 2015). In this case, the accuracy is guaranteed by the fact $\|e^{\text{logm}(\sigma_{r_0})} - \sigma_{r_0}\|_1 \leq 10^{-16}$ and $\|e^{\text{logm}(\sigma_0)} - \sigma_0\|_1 \leq 10^{-14}$. Noting that the difference between $2R(\rho_{r_0})$ and $E^+_R(\rho_{r_0}^2)$ is already $1.00691 \times 10^{-2}$, we can safely claim that

$$R(\rho_{r_0}^2) \leq E_R(\rho_{r_0}^2) \leq E^+_R(\rho_{r_0}^2) < 2R(\rho_{r_0}).$$

It is also easy to observe that

$$E^\infty_R(\rho_{r_0}) \leq \frac{1}{2}E_R(\rho_{r_0}^2) < R(\rho_{r_0}).$$
Rains’ bound is not additive (cont.)

When $0.45 \leq r \leq 0.548$, we show the gap between $2R(\rho_r)$ and $E_R^+(\rho_r \otimes^2)$ in the following figure:

![Figure](image.png)

**Figure:** This plot demonstrates the difference between $2R(\rho_r)$ and $E_R^+(\rho_r \otimes^2)$ for $0.45 \leq r \leq 0.548$. The dashed line depicts $E_R^+(\rho_r \otimes^2)$ while the solid line depicts $2R(\rho_r)$. 

Xin Wang and Runyao Duan, AQIS'2016
Applications and future directions

- Regularization of Rains’ bound:
  \[ R_{\infty}(\rho) = \inf_{k \geq 1} \frac{R(\rho \otimes^k)}{k}. \]

- A better upper bound on distillable entanglement:
  \[ E_\Gamma(\rho) \leq R_{\infty}(\rho) \leq R(\rho), \]
  and the second inequality could be strict.

- **Remarks:** We were informed by one of AQIS’16 referees that Hayashi introduced the regularization of Rains’ bound in his book in 2006.

- How about \( R_{\infty} \) and \( E_R^\infty \)?

  \[ R_{\infty}(\rho) < E_R^\infty(\rho) \text{ for any rank-2 mixed state } \rho \text{ supporting on the anti-symmetric } 3 \otimes 3 \text{ space, thus the irreversibility of PPT manipulation of entanglement, the 20}\text{th open problem in Quantum Information Theory on Werner’s website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).} \]
Applications and future directions

- Regularization of Rains’ bound:

\[ R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho \otimes^k)}{k}. \]

- A better upper bound on distillable entanglement:

\[ E_\Gamma(\rho) \leq R^\infty(\rho) \leq R(\rho), \]

and the second inequality could be strict.

- Remarks: We were informed by one of AQIS’16 referees that Hayashi introduced the regularization of Rains’ bound in his book in 2006.

- How about \( R^\infty \) and \( E_R^\infty \)?

- \( R^\infty(\rho) < E_R^\infty(\rho) \) for any rank-2 mixed state \( \rho \) supporting on the anti-symmetric 3 \( \otimes \) 3 space, thus the irreversibility of PPT manipulation of entanglement, the 20\(^{th}\) open problem in Quantum Information Theory on Werner’s website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).
Applications and future directions

- Regularization of Rains’ bound:
  \[ R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho \otimes^k)}{k}. \]

- A better upper bound on distillable entanglement:
  \[ E_\Gamma(\rho) \leq R^\infty(\rho) \leq R(\rho), \]
  and the second inequality could be strict.

- **Remarks:** We were informed by one of AQIS’16 referees that Hayashi introduced the regularization of Rains’ bound in his book in 2006.

- How about \( R^\infty \) and \( E^\infty_R \)?

- \( R^\infty(\rho) < E^\infty_R(\rho) \) for any rank-2 mixed state \( \rho \) supporting on the anti-symmetric \( 3 \otimes 3 \) space, thus the irreversibility of PPT manipulation of entanglement, the 20\(^{th}\) open problem in Quantum Information Theory on Werner’s website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).
Applications and future directions

- Regularization of Rains’ bound:

\[ R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho^\otimes k)}{k}. \]

- A better upper bound on distillable entanglement:

\[ E^\Gamma(\rho) \leq R^\infty(\rho) \leq R(\rho), \]

and the second inequality could be strict.

- **Remarks:** We were informed by one of AQIS’16 referees that Hayashi introduced the regularization of Rains’ bound in his book in 2006.

- How about \( R^\infty \) and \( E^\infty_R \)?

- \( R^\infty(\rho) < E^\infty_R(\rho) \) for any rank-2 mixed state \( \rho \) supporting on the anti-symmetric \( 3 \otimes 3 \) space, thus the irreversibility of PPT manipulation of entanglement, the 20th open problem in Quantum Information Theory on Werner’s website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).
Applications and future directions

- Regularization of Rains’ bound:
  \[ R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho^\otimes k)}{k}. \]

- A better upper bound on distillable entanglement:
  \[ E_\Gamma(\rho) \leq R^\infty(\rho) \leq R(\rho), \]
  and the second inequality could be strict.

- **Remarks:** We were informed by one of AQIS’16 referees that Hayashi introduced the regularization of Rains’ bound in his book in 2006.

- How about \( R^\infty \) and \( E^\infty_R \)?

- \( R^\infty(\rho) < E^\infty_R(\rho) \) for any rank-2 mixed state \( \rho \) supporting on the anti-symmetric 3 \( \otimes \) 3 space, thus the irreversibility of PPT manipulation of entanglement, the 20\(^{th}\) open problem in Quantum Information Theory on Werner’s website, proposed by Plenio in 2005. (See Xin Wang and Runyao Duan, arXiv:1606.09421).