Dimension witnesses beyond non-classicality tests

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Motivation – High d Systems

Generation and confirmation of a (100 × 100)-dimensional entangled quantum system

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Author Affiliations

Contributed by Anton Zeilinger, February 24, 2014 (sent for review December 15, 2013)

Direct measurement of a 27-dimensional orbital-angular-momentum state vector

Mehul Malik, Mohammad Mirhosseini, Martin P. J. Lavery, Jonathan Leach, Miles J. Padgett & Robert W. Boyd

Affiliations | Contributions | Corresponding author

_Nature Communications_ 5, Article number: 3115 | doi:10.1038/ncomms4115
Received 22 October 2013 | Accepted 16 December 2013 | Published 20 January 2014
Dimension Witnesses

Testing the Dimension of Hilbert Spaces
Nicolas Brunner, Stefano Pironio, Antonio Acín, Nicolas Gisin, André Allan Méthot, and Valerio Scarani
Phys. Rev. Lett. 100, 210503 – Published 30 May 2008

Lower bound on the dimension of a quantum system given measured data
Stephanie Wehner, Matthias Christandl, and Andrew C. Doherty
Phys. Rev. A 78, 062112 – Published 22 December 2008

Dimension Witnesses and Quantum State Discrimination
Nicolas Brunner, Miguel Navascués, and Tamás Vértesi
Phys. Rev. Lett. 110, 150501 – Published 8 April 2013

Device-independent dimension tests in the prepare-and-measure scenario
Jamie Sikora, Antonios Varvitsiotis, Zhaohui Wei
(Submitted on 13 Jun 2016)
Our Result

Assuming the dimension of a physical state is \( d \), there is a witness that distinguishes between fully quantum and classical states, including separable partitions.

E.g. let \( d = ab \), then the witness should distinguish between

\[ Q_{ab}, Q_a Q_b, C_a Q_b, C_b Q_a, C_{ab} \]
Random Access Codes (RACs)

- **Input for Alice**
  \[ x = x_0 x_1 ... x_{n-1} \quad ; \quad x_i \in \{0,1,\ldots,d - 1\} \]

- **Input for Bob**
  \[ y \in \{0,1,\ldots,n - 1\} \]

- **Rules (Prepare & Measure)**
  Alice prepares and sends a \(d\)-dimensional state to Bob. Bob measures the state and outputs \(b \in \{0,1,\ldots,d - 1\}\).

- **Success Condition**
  \[ b = x_y \]

- **Figure of Merit**
  Average Success Probability
  \[
  \bar{p} = \frac{1}{nd^n} \sum_{x,y} p(b = x_y | X = x, Y = y) = \frac{1}{n} \sum_y p(\text{Correctly guessing } x_y)
  \]
$2^2 \rightarrow 1$ QRAC ; An Example

In general, for $2^d \rightarrow 1$ RACS:

$$\bar{p}_q = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{d}} \right)$$

$$\bar{p}_c = \frac{1}{2} \left( 1 + \frac{1}{d} \right)$$

For $d=2$,
$$\bar{p}_c = 0.75 \ , \ \bar{p}_q \approx 0.854$$

Ambainis et al., Quantum Random Access Codes with Shared Randomnes, q-ph: 0810.2937
Our Framework and non-Adaptive Strategies

- Let $d = d^0 d^1 \cdots d^{k-1}$, and suppose we want to know the maximum $\bar{p}$, for $Q_{d^0} Q_{d^1} \cdots Q_{d^{k-1}}$

- Alice sends $|\psi\rangle = |\psi^0\rangle |\psi^1\rangle \cdots |\psi^{k-1}\rangle$ to Bob.

- Bob measures each qudit with projective measurements.

- In general, the choice of the $j$th measurement basis could depend on all previous measurement outcomes. We call this an Adaptive Strategy.

- **Lemma:** For an Adaptive Strategy there exists a non-Adaptive Strategy which has at least the same average success probability.
Magic Curves for $2^d \rightarrow 1$ QRACs

- $m_d(x) = \cos \left( \cos^{-1} \left( \frac{1}{\sqrt{d}} \right) - \cos^{-1}(\sqrt{x}) \right)^2 = m_d(x) = 1 - \left( \frac{d-1}{d} \right) \left( \sqrt{x} - \sqrt{\frac{1-x}{d-1}} \right)^2$
The Magic...

- $\bar{p} = \frac{1}{2} (p(\text{correctly guessing 0th dit}) + p(\text{correctly guessing 1st dit}))$

- Now, e.g., let $d = ab$, then write:
  $x_0 = x_0^0 x_0^1$, $x_1 = x_1^0 x_1^1$ with $x_0^0 \in \{0, 1, \ldots, a - 1\}$, $x_1^1 \in \{0, 1, \ldots, b - 1\}$

- $u = p(\text{correctly guessing FIRST part of 0th dit})$
  $v = p(\text{correctly guessing SECOND part of 0th dit})$

\[
\bar{p} = \frac{1}{2} \max_{u,v} (uv + m_a(u)m_b(v))
\]

- $d=2*2$
- $d=8*8$
Our Result Applied to Examples

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Questions?