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Encoding a Qubit into an Oscillator using Phase Estimation

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Abstract. In 2001 Gottesman, Preskill and Kitaev proposed to encode a qubit into an oscillator (bosonic mode) using so-called comb or grid states. We show how such states can be useful as displacement sensors, going beyond squeezed states. We present new protocols for preparing these states using a dispersive interaction with a qubit (as is common in circuit-QED) or using cat states, beam-splitters and homodyne detection. Our protocols largely bypass the need for post-selection (which had been the stumbling block in the literature until now) as we can interpret the protocols as a sequential implementation of phase estimation.

Fault-Tolerant Error Correction for non-Abelian Anyons

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Abstract. While topological quantum computation is intrinsically fault-tolerant at zero temperature, it looses its topological protection at any finite temperature. We present a scheme to protect the information stored in a system supporting non-cyclic anyons against thermal and measurement errors [1]. The correction procedure builds on the work of Gács [2] and Harrington [3] and operates as a local cellular automaton. In contrast to previously studied schemes, ours is valid for both abelian and non-abelian anyons and accounts for measurement errors. We prove the existence of a fault-tolerant threshold and numerically simulate the procedure for Ising anyons. Our simulations are consistent with a threshold between 10^{-4} and 10^{-3}.

Keywords: non-abelian anyons, fault-tolerant error correction, cellular automaton, quantum memory

1 Background

Anyons and quantum computation. Non-abelian anyons are hypothetical particles with very exotic properties that defy intuition but that are nonetheless permitted by known laws of physics. These particles have drawn much interest due to their suspected existence in two-dimensional condensed matter systems and for their potential applications in quantum computation [4]. In particular, a quantum computation can in principle be realized by braiding and fusing certain non-abelian anyons. These operations are expected to be intrinsically robust due to their topological nature.

Systems supporting anyonic excitations have a spectral gap Δ. Provided the system is kept at a temperature T lower than the spectral gap, the density of thermal excitations is suppressed by an exponential Boltzmann factor $e^{-\Delta/T}$. However, this thermal protection is not scalable: thermal excitations do appear at constant density for any non-zero temperature and so their presence is unavoidable as the size of the computation increases. Thermally activated anyons can corrupt the encoded data by braiding or fusing with the computational anyons. It thus appears necessary to supplement topological quantum computation with some form of quantum error correction.

Error correction for abelian anyons. Error correction in abelian anyonic models is intrinsically linked to topological quantum error correction with the toric code and has thus been studied extensively. There, it is possible to model the different thermal processes phenomenologically using a particle creation rate and a diffusion rate. Error correction monitors the presence of these thermal excitations by periodically measuring the topological charge at every lattice site. A *decoding algorithm* is used to statistically infer the homology of each particle's world-line from these snapshots, thus enabling the recovery of the topological information. It is now well established that these systems possess a threshold: below a critical "temperature", the logical error rate can be suppressed to arbitrarily low values by increasing the system size [5].

These error-correction studies assume that the topological charge measurements are perfect. In a realistic setting, a measurement can report the wrong charge—e.g., report a charge when the site if empty or fail to report a charge—and moreover it can introduce additional errors. The ability to protect a topologically ordered systems using such noisy charge measurements is intrinsically linked to fault-tolerant topological quantum error correction, where fault-tolerance refers to the ability to combat errors with noisy instruments. Again, for abelian anyons, this problem has been studied extensively and is known to possess a fault-tolerance threshold.

Error correction for non-abelian anyons. The theory of error correction for non-abelian anyons is in contrast far less developed. Specific examples of error correction algorithms for Ising anyons [6], the $\Phi - \Lambda$ model [7] and Fibonacci anyons [8] have been investigated numerically and found to posses a threshold. Additionally, greedy hard-decision renormalization group decoders can error-correct any systems giving rise to anyonic excitations [9, 10]. However, none of these studies have considered the case where the charge measurements are faulty, a serious complication for all the previous methods.

2 Main Ideas

The decoding algorithm. The basic idea of all topological decoding algorithm is to pair up the thermal anyons, bring the anyons of a pair together and hope that they fuse to the vacuum. The details of how the pairs are chosen and how the anyons of a pair are brought together is specific to each decoding algorithm.

Gács' proof that one-dimensional cellular automatons can process information in a fault-tolerant way is notoriously complex. Harrington builds on this proof by explaining how key concepts need to be adapted to the toric code setting. Similarly, our proof builds on Harrington's proof and focuses on the key novelties introduced by the non-abelian nature of the anyons.

The key idea in Gács and Harrington's approach is to classify errors into distinct groups called actual errors characterized by a level, labeled by an integer k. Roughly, the level of an actual error describes its spatial extension

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and separation from other actual errors, both exponentially increasing with k. The idea is then to demonstrate that level-k errors are effectively suppressed at the kth level of renormalization of the decoding procedure. In a sense, errors of different levels do not interact with each other and can be analyzed independently. The non-trivial braiding relations and fusion rules of non-abelian anyons break this simple structure.

Complications with non-abelian anyons. Non-abelian anyons present novel obstacles to this general decoding strategy. First, the fusion process is intrinsically irreversible for non-abelian anyons. In particular, when two anyons of 'opposite' topological charges a and $\bar{a}$ are brought together, they may fuse to a non-trivial charge (*i.e.* a left-over excitation remains). Observing the outcome of this fusion is an irreversible process (collapse of the wave-function), so the error-correction process itself could introduce physically irreversible changes to the system. Second, there are many more ways in which small errors involving non-abelian anyons can build up to a larger error. The non-abelian nature of the excitations also presents significant additional obstacles to the analysis of the error correction procedure itself.

The fusion-rules of non-abelian anyons are in general non-deterministic. As in Harrington's approach, we show that the level-k syndrome will correctly identify the topological charge of a renormalized level-k cell. But even when a particle and its anti-particle have been correctly identified, their fusion may result in a non-trivial particle. As a consequence, our proof needs to apply to all possible fusion histories of the anyons. For this purpose, we introduce the notion of the trajectory domain of an error, which is roughly the set of sites that have a finite amplitude of becoming charged as a consequence of a given error.

The non-abelian braiding relations have deeper consequences on the error classification. Consider a situation where a low-level actual error E is well isolated, in space and time, from any other actual error. The low-level correction rules tend to concentrate all the excitations caused by E onto a single site, which will result in the vacuum and thus the elimination of E. Suppose however that a high-level transition rule drags an anyon through the region containing E. This is not forbidden by the definition of level-k actual errors: those need to be well isolated from each other and from higher-level errors, but this does not prevent high-level transition rules from operating in their vicinity. As a consequence of the non-abelian braiding rules, after the passage of the high-level anyon, the excitations created by E may no longer fuse to the vacuum: they have become entangled with the high-level error. Thus, neither the error E nor the higher-level error with which it has become entangled can be corrected individually: they need a joint correction strategy.

Existence of a threshold for non-cyclic anyons. This entanglement across errors of different levels requires the definition of causally-linked clusters of errors. These are collections of actual errors of distinct levels that have potentially become entangled through the transition rules applied by the correction algorithm. Despite this new failure mechanism, repeated applications of the correction rules are bound to succeed for non-cyclic anyon models, a family of non-abelian anyons we introduce in this work. Indeed, every failed attempt moves the total charge of the anyon which is dragged by the high-level transition rule closer to becoming abelian. After a finite number of attempts, the topological charge will become abelian, so the next iteration is guaranteed to succeed as in an abelian model. Thus, the net effect is a possible slow-down of the correction process, which can be compensated by a lower error threshold. Numerical simulations of Ising anyons are consistent with a threshold for the error rate in the range of $10^{-4} \sim 10^{-3}$.

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Quantum noise spectroscopy

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Abstract. We present spectroscopy protocols capable of characterizing the noise affecting a set of qubits. Concretely, we are able to reconstruct the correlation functions of the bath in frequency space, i.e., the power spectra of the environment. This is achieved by measuring the response of the qubits under control pulses with particular symmetries and in the presence of the bath of interest. We discuss implications of these protocols to quantum control, metrology, and fault-tolerant quantum computing.

Keywords: Quantum Control, Noise Spectroscopy

Quantum systems, and in particular qubits, are the core of the so called *quantum technologies*, which range from quantum sensors to quantum computers. Central to the implementation of these technologies is the ability to control quantum systems with high precision, in the presence of undesired interactions with external, classical or quantum, degrees of freedom, i.e., the environment. Active control methods to achieve this, such as Quantum Error Correction, Optimal Control or Dynamical Decoupling [1], have been proposed and experimentally demonstrated during in recent years. Such methods can be roughly divided in two categories: (i) methods that use specific knowledge of the noise, but are typically very susceptible to uncertainty in such knowledge, to design efficient protocols, and (ii) methods that are effective with minimal assumptions on the noise, but are generally not efficient and do not scale well with the size of the system. Clearly, the ideal scenario is that in which complete knowledge of the noise is available, but this is generally not the case.

Mathematically, all information about the evolution of a quantum state of a system coupled to a bath, $\rho_{SB}(t)$, can be determined from its initial state, assumed here to be factorisable for simplicity, and the Hamiltonian:

$$\rho_{SB}(0) = \rho_S(0) \otimes \rho_B(0) \text{ and } H(t).$$

Now, in general, the bath is inaccessible and unmeasurable, and only the system is what is of interest. As such one is interested in predicting and controlling the evolution of $\rho_S(t) = \mathrm{tr}_B[\rho_{SB}(t)]$. Since, in general, $H(t) = H_S + H_B + H_{SB}$ includes both the system and the bath self dynamics (H_S, H_B) as well as an interaction term H_{SB}, then

$$\rho_S(t) = \phi(\rho_S(0)),$$

is described by a decoherence-inducing CPTP map $\phi(\cdot)$. If knowledge of $\rho_S(0)$, $\rho_B(0)$, and $H(t)$ is available, then it is in principle straightforward to predict the dynamics of the state, $\rho_S(t)$, by explicitly constructing the map $\phi(\cdot)$, using Master equation techniques or Path Integral methods for example [2]. What is more, one could also use any of the powerful optimal control ((i) above) methods to design control Hamiltonians capable of efficiently minimizing the decoherence induced by the coupling between the system and the bath to a desired level.

Unfortunately, $\rho_B(0)$ and $H(t)$ are rarely fully known, which is only natural given that, in general, they cannot be directly measured in general. Phenomenological arguments can shine light one them but there is always a degree of uncertainty. How to move forward, then?

The key observation is that the dynamics of a system due to its interaction with a external (typically inaccessible and uncontrollable) degrees of freedom, i.e., the bath, can always be written [6, 7] in terms of convolutions between the filter functions, i.e., purely control dependent functions, and the *power spectra* of the bath, i.e., the Fourier transform of the correlation functions of bath operators. Let us make this statement more explicit. One can always write the Hamiltonian of system and bath (in an appropriate interaction picture) via

$$H = \sum_{a,b} y_{a,b,}(t) Q_a \otimes B_b(t),$$

where $\{Q_a\}$ is an operator basis for the system Hilbert space, $B_b(t)$ is a time-dependent bath operator, and $y_{a,b,}(t)$ si switching function that contains the effect of the control. One can show that $\rho_S(t)$ is a function of convolutions of the form

$$\int_{-\infty}^{\infty} d\omega_1 \cdots d\omega_k F^{(k)}_{a_1,\cdots;b_1,\ldots}(\omega_1,\cdots,\omega_k,T) S^{(k)}_{b_1,\cdots,b_k}(\omega_1,\cdots,\omega_k),$$

where the filter function

$$F^{(k)}_{\vec{a},\vec{b}}(\vec{\omega},T) = \int_0^T ds_1 \cdots \int_0^{s_{k-1}} ds_k y_{a_1,b_1}(s_1)\cdots y_{a_k,b_k}\cdots e^{i\vec{\omega}\cdot\vec{s}}$$

is fully known, as depends on the control only, and the power spectra $S^{(k)}_{b_1,\cdots,b_k}(\omega_1,\cdots,\omega_k)$ is the Fourier transform of the bath correlator

$$\mathrm{Tr}[B_{b_1}(s_1)\cdots B_{b_k}(s_k)\rho_B].$$

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The above implies that there is no need to know ρ_B and $\{B_b(t)\}$, but only their interplay in the bath correlation functions.

Given the above, it is clear that being able to access information about the correlations in the bath would be an important tool in the quantum control arsenal. A few applications come to mind if *detailed knowledge of all the power spectra of the bath affecting a set of qubits* was available:

- The problem of having high quality quantum systems to execute a particular protocol would be one of designing controls, and thus the filters, capable of minimizing the value of the most relevant convolutions within fixed experimental constraints, e.g., bandwidth.
- Applications that use the bath as resource, e.g., to generate entanglement between qubits [8], could be fully controlled, optimized, etc.
- If, additional information about the environment was available, physical parameters could be extracted [9, 10]. For example, the temperature of a bosonic thermal environment.
- Questions related to the rate of decay of correlations, crucial for fault-tolerant quantum computing applications [11], could be addressed.

This has motivated a recent push [3, 4, 5] to develop *Noise spectroscopy protocols* that use a qubit as a probe to characterize the underlying noise process affecting it, in terms of the bath correlation functions. Many of such protocols have been proposed and experimentally demonstrated. Initial efforts assumed a functional form (with some free parameters) for the power spectra [3]. This facilitated the deconvolution of the relevant integrals and allowed the use of the measured response of the qubit to estimate the parameters. The success of such protocols heavily depended on the accuracy of the assumed knowledge and this was a limitation. More recently, protocols that can reconstruct an 'arbitrary' (within certain constraints we discuss below) power spectra have been demonstrated [4, 5]. Their key contribution is the idea that symmetries in the control can lead to a way to deconvolve the relevant integrals and thus avoids the need to make a assumptions on the power spectra. However, they have been so far limited to the case of Gaussian noise, i.e., when only one power spectra is non-vanishing, affecting a single qubit. Moving beyond these limitations is imperative if characterizing the bath affecting systems relevant to quantum information processing applications is the main objective: we need to be able to characterize general noise affecting multiple quits.

In this talk I will discuss advancement we have made in this direction, focusing mainly on two results:

Non-Gaussian noise spectroscopy [12].- The Gaussian noise assumption is valid in various physical systems in the weak coupling or the short-time limits, or is naturally satisfied for certain specific cases, e.g., a bosonic thermal environment. However, beyond these scenarios non-Gaussian noise is the rule, e.g., $1/f$ noise in superconducting qubits.

We show that, by exploiting the mathematical structure of filter functions recently explored [7], it is possible to (i) extend the sampling range current Gaussian noise spectroscopy beyond the 'expected' limit set by the inverse of the switching time, and (ii) to reconstruct the higher order power spectra characterizing a classical non-Gaussian noise source such as random telegraph noise. We demonstrate the success of the protocol via numerical examples.

Multiqubit noise spectroscopy [13].- Since one is typically interested in multiple qubits for quantum technology applications, there is a need to characterize the noise affecting multiple qubits. Here by exploiting and combining two types of symmetries in the control (antisymmetry and repetition), we show that one can reconstruct all the power spectra describing the action of Gaussian noise (or approximately Gaussian) on a set of qubits. Concretely, a model described by a Hamiltonian of the form

$$H(t) = \sum_{i=1}^{N} \sigma_z^{(i)} \otimes B_i(t) + \sum_{i,j} \sigma_z^{(i)} \otimes \sigma_z^{(j)} \otimes B_{i,j}(t), \quad (1)$$

where the generic bath operators $\{B_i(t)\}$ have Gaussian statistics (or to a good approximation). That is, they are such that

$$C^{(k)}(B_{i_1}(t_1), B_{i_2}(t_2) \cdots B_{i_k}(t_k)) = 0 \text{ for } k > 2,$$

where $C^{(k)}(B_{i_1}(t_1), B_{i_2}(t_2) \cdots B_{i_k}(t_k))$ is the k-th cumulant [14] of $\{B_{i_1}(t_1), B_{i_2}(t_2), \cdots, B_{i_k}(t_k)\}$. A bosonic thermal environment with linear coupling satisfies this condition exactly and we demonstrate the execution of our protocol in such model.

Finally we will expand on the implications of having access to such knowledge and some of the important applications discussed earlier in this abstract, particularly optimized entanglement generation, bath thermometry and characterizing the decay bath correlations for fault-tolerant quantum computing.

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Decoupling with Random Diagonal-Unitaries

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Abstract. Decoupling is one of the most important primitives in a wide range of quantum science. It is known that approximate unitary 2-designs can achieve decoupling at a certain rate if the degree of approximation is sufficiently precise. Here, we propose a new scheme of achieving decoupling, based on repeated applications of a random diagonal-unitary in the Pauli-X basis and that in the Pauli-Z basis, and show that decoupling can be achieved by rather imprecise approximate unitary 2-designs. We also provide a simple quantum circuit for decoupling and show implications on two applications.

Keywords: Decoupling, Unitary design, Quantum Shannon theory, Quantum thermodynamics

1 Introduction and main results

The task of decoupling is to destroy all possible correlations between two systems by applying a unitary to one of the systems (see Fig 1 for the full description). This task is playing significant roles in quantum Shannon theory, as it can be used to prove most of the capacity theorems by showing the existence of a decoder, and also in quantum thermodynaics, in the blackhole physics, and quantum many-body physics, where fundamental phenomena can be understood as consequences of decoupling.

Decoupling was originally studied using Haar random unitaries, unitaries drawn from a unitary group uniformly at random according to the Haar measure, and was shown to be achievable at a certain rate [1]. It is more important to investigate decoupling with unitary t-designs, which simulates up to the tth order moments of Haar random unitaries, since Haar random unitaries cannot be efficiently implemented. Simply due to the fact that decoupling uses at most the second order properties of Haar random unitaries, exact unitary 2-designs can trivially achieve decoupling at the same rate as the Haar one. Exact designs are however not necessary because even Haar random unitaries cannot decouple the system perfectly. Motivated by this, decoupling with approximate unitary 2-designs was studied, and it turned out that decoupling can be achieved if the degree of approximation is sufficiently precise [2].

It is however still open whether the precision of approximate unitary 2-designs in Ref. [2] is *necessary or not* to achieve decoupling at the same rate as the Haar random one. Addressing this question is practically and theoretically important because approximate unitary 2-designs with less precisions are presumably easier to implement, and answering this question provides some insight to another open problem of whether unitary 2-designs are really needed in decoupling.

In this work, we provide a new construction of decoupling [3], where random unitaries diagonal in the Pauli-X and those in the Pauli-Z bases are applied alternately and repeatedly. By showing that the process achieves decoupling at the same rate as the Haar one but cannot be a precise unitary 2-design [4], we reach our conclusion that the precision of approximate unitary 2-designs in Ref. [2] is not necessary to achieve decoupling. Moreover, when the CPTP map $\mathcal{T}_{A\to B}$ (see Fig. 1) is the partial trace over a subsystem in A, which is the most important case in applications of decoupling, we also show that decoupling can be achieved with even less precise approximate unitary 2-designs.

We then provide a quantum circuit implementing decoupling using $O(N_A^2)$ two-qubit gates, where N_A is the number of qubits in the system A (see Fig. 1). In terms of the number of gates, this circuit is unfortunately less efficient than the best known result using $O(N_A \log N_A)$ gates, but our construction has a couple of advantages. One is that the circuit is divided into a constant number of commuting parts and all gates in each commuting part can be applied simultaneously. The other is that, from the physical point of view, the circuit corresponds to the Hamiltonian dynamics where the interactions are two-body and are changed only a few times, independent of the system size. This "nearly time-independent" feature is unique to our construction and is important in the understanding of a natural mechanism that leads to decoupling in many-body systems.

We also explain implications on physical and information theoretic tasks, such as *relative thermalization* [5] and the *state merging protocol* [6].

2 Brief descriptions of our theorems

The *decoupling* protocol is explained in Fig. 1. A *unitary t-design* is defined as follows; for a bounded operator X on the t-copies of a Hilbert space $\mathcal{H}$, let $\mathcal{G}_U^{(t)}(X) := \mathbb{E}_U[U^{\otimes t} X U^{\dagger\otimes t}]$ where $\mathbb{E}_U$ represents an expectation over a random unitary U. A δ-approximate unitary t-design is a random unitary U satisfying $\|\mathcal{G}_U^{(t)} - \mathcal{G}_{U^H}^{(t)}\|_\diamond \leq \delta$ where U^H is a Haar random unitary.

It was shown that the decoupling rate $\Lambda_{\rm Haar}$ (see the

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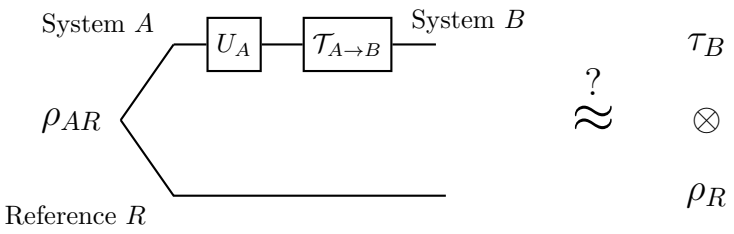

Figure 1: The decoupling protocol [1]. The system A of the initial state ρ_{AR} goes through a unitary evolution U_A and a CPTP map $\mathcal{T}_{A\to B}$. The goal of decoupling is to make the output as close to $\tau_B \otimes \rho_R$ as possible by choosing an appropriate unitary U_A, where τ_B is a marginal state in B of a Jamiołkowski state of $\mathcal{T}_{A\to B}$, and $\rho_R = \mathrm{tr}_R \rho_{AR}$. When U_A is a Haar random unitary, the output state is Λ_{Haar}-close to $\tau_B \otimes \rho_R$ with high probability, where $\Lambda_{\mathrm{Haar}} = 2^{-\frac{1}{2}(H^\epsilon_{\min}(A|R)_\rho + H^\epsilon_{\min}(A|B)_\tau)} + 12\epsilon$ and $H^\epsilon_{\min}$ is the ϵ-smooth conditional min-entropy [1].

caption of Fig. 1) is achieved if U is a Haar random unitary or an exact unitary 2-design [1]. Moreover, δ-approximate unitary 2-designs can also achieve the decoupling rate $O(\Lambda_{\mathrm{Haar}})$ if $\delta = O(1/d_A^4)$, where d_A is the dimension of the system A [2]. Our goal is to show that this precision of approximate unitary 2-designs is not necessary to achieve the decoupling rate $O(\Lambda_{\mathrm{Haar}})$.

To state our main results, let D^W be a random W-diagonal-unitary ($W = X, Z$), which is a diagonal unitary in the Pauli-W basis with random phases. We then introduce a random unitary $D[\ell] := D^Z_{\ell+1} D^X_\ell D^Z_\ell \cdots D^X_1 D^Z_1$, where each D^W_i is independent.

Theorem 1 (Ref. [4]) *The random unitary $D[\ell]$ on an N-qubit system is a δ-approximate unitary 2-design for $\ell \geq 2 + \frac{1}{N}(1 + \log 1/\delta)$. Conversely, $D[\ell]$ cannot be a δ-approximate unitary 2-design if $\ell \leq \frac{1}{N} \log 1/\delta$.*

It follows from Theorem 1 and Ref. [2] that decoupling at the rate $O(\Lambda_{\mathrm{Haar}})$ is achieved by $D[\ell]$ if $\ell \geq 7$. However, we show that $\ell \geq 3$ is sufficient for achieving decoupling at the rate $O(\Lambda_{\mathrm{Haar}})$

Theorem 2 (Ref. [3]) *Let $\ell \geq 2$ and $\epsilon > 0$. Then,*

$$\mathbb{E}_{U_A \sim \mathrm{D}[\ell]} \| \mathcal{T}_{A\to B}(U_A \rho_{AR} U_A^\dagger) - \tau_B \otimes \rho_R \|_1 \leq \Lambda_{D[\ell]}, \quad (1)$$

where $\Lambda_{D[\ell]} = \sqrt{c_\ell} 2^{-\frac{1}{2}(H^\epsilon_{\min}(A|R)_\rho + H^\epsilon_{\min}(A|B)_\tau)} + 12\epsilon$, $c_\ell = 2 + O(d_A^{-(\ell-3)})$, and d_A is the dimension of A.

We immediately obtain the following corollary, which provides our main statement that precise approximate unitary 2-designs are not necessary for decoupling.

Corollary 3 *The random unitary $D[\ell]$ achieves decoupling at the rate $O(\Lambda_{\mathrm{Haar}})$ if $\ell \geq 3$. When $\ell = 3$, the $D[\ell]$ cannot be a $O(1/d_A^4)$-approximate unitary 2-design.*

Furthermore, we can also show that $D[\ell]$ with $\ell = 2$ achieves decoupling when the CPTP map $\mathcal{T}_{A\to B}$ is given by the partial trace of a subsystem of A. This case is particularly important in most applications of decoupling.

Using the result in Ref. [7], we also provide a quantum circuit that implements $D[\ell]$, and so, achieves unitary 2-designs and decoupling. The circuit repeats the following steps $O(\ell)$ times; single-qubit phase gates on all qubits with phases randomly chosen from $\{0, 2\pi/3, 4\pi/3\}$, the controlled-Z gates acting on every pair of qubits with probability 1/2, and the Hadamard gates on all qubits. In total, the circuit uses $O(\ell N^2)$ gates.

3 Discussions

Our results indicate the possibility that random unitaris strictly less uniform than unitary 2-designs, not in the sense of approximation, could achieve decoupling at the rate $O(\Lambda_{\mathrm{Haar}})$. This would also lead to the extension of unitary t-designs to non-integer t, because only 1-designs are less uniform than 2-designs in the current framework, but they cannot achieve decoupling. It would be interesting to introduce t-designs with non-integer t and investigate decoupling with them, which may be done using the *frame potential* or by the direct investigation based on continuous Hamiltonian dynamics.

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