Quantum centrality ranking via quantum walks and its experimental realisation

Joshua Izaac$^1$, Xiang Zhan$^2$, Jian Li$^2$, Peng Xue$^2$, Paul Abbott$^1$, Xiaosong Ma$^3$ and Jingbo Wang$^1$
Single Particle undirected CTQW

\( A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \)

\( H_{ij} = \left( \sum_k A_{ik} \right) \delta_{ij} - A_{ij} \)

\( H = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix} \)

\( \hat{U}(t) = e^{-iHt} = \exp \left( -it \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix} \right) \)

\[ \frac{d}{dt} |\psi(t)\rangle = -iH|\psi(t)\rangle \]
Centrality measures

A graph centrality measure $C$ satisfies the following properties:

1. $C : G(V, E) \to \mathbb{R}^{|V|}$ is a function or algorithm that accepts a graph and returns a real-valued and strictly positive vector over the set of vertices $V$

2. Higher values are provided to vertices deemed more ‘important’ or ‘central’ to the graph structure
Classical centrality

The Eigenvector centrality is given by

\[ C_{j}^{ev} = v_{j} \quad \text{where} \quad A v = \lambda v \]

It can be shown that

\[ v_{j} \propto \sum_{i=1}^{N} \sum_{k=1}^{\infty} \lambda^{1-k} (A^{k})_{ii} \]

\[ U(t)_{j} = (e^{-iHt})_{j} = \sum_{i=1}^{N} \sum_{k=1}^{\infty} \frac{(-it)^{k}}{k!} (H^{k})_{ij} \]
Quantum Centrality

- **DTQW with frequency analysis**

- **Quantum PageRank**
  Paparo and Martin-Delgado (2012)

- **Quantum Stochastic Walk**
  Falloon, Rodriguez, and Wang (2016)

- **CTQW-based Jensen-Shannon divergence**
  Rossi (2014)

The ability to physically realize these quantum centrality measures is currently beyond our reach.
CTQW Quantum Centrality

1. Start in an equal superposition of vertex states

2. Propagate for time $t$

3. Find the probability of the walker on vertex $j$ at time $t$

4. Calculate the long-time average probability of locating the walker at vertex $j$

$$C_j^{(CTQW)} = \lim_{\tau \to \infty} \frac{1}{\tau} \left| \langle j | e^{-iHt} \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \right) \right|^2 dt$$
\[ T = \frac{\pi}{\sqrt{3}} \]

\[ \langle 0 \rangle = \frac{1}{2} \]

\[ \langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = \frac{1}{6} \]
Erdős-Rényi $G(20,0.3)$

<table>
<thead>
<tr>
<th>Degree</th>
<th>PageRank</th>
<th>Eigenvector</th>
<th>CTQW</th>
<th>RWC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>1.</td>
<td>0.882</td>
<td>0.882</td>
<td>0.882</td>
</tr>
<tr>
<td>PageRank</td>
<td>1.</td>
<td>0.622</td>
<td>0.622</td>
<td></td>
</tr>
<tr>
<td>Eigenvector</td>
<td>1.</td>
<td>1.</td>
<td></td>
<td>0.87</td>
</tr>
<tr>
<td>CTQW</td>
<td>1.</td>
<td>0.87</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RWC</td>
<td>1.</td>
<td></td>
<td></td>
<td>0.87</td>
</tr>
</tbody>
</table>
The average CTQW centrality measure (black) compared to classical centrality measures for vertices in an ensemble of 200 Erdos-Renyi graphs $G(100,0.3)$. 

\[ \tau = 0.843 \pm 0.020 \]

\[ \tau = 0.991 \pm 0.004 \]
Experimental implementation

\[
\begin{align*}
\Delta \delta &= \frac{9}{40} \approx \frac{\pi}{8} = \frac{T}{8} \\
U(\Delta t) &= \\
&= \begin{pmatrix}
0.925019 & 0. -0.219348i & 0. -0.219348i & 0. -0.219348i \\
0. -0.219348i & 0.975006 & -0.0249938 & -0.0249938 \\
0. -0.219348i & -0.0249938 & 0.641673 & -0.0249938 \\
0. -0.219348i & -0.0249938 & -0.0249938 & 0.975006
\end{pmatrix}
\end{align*}
\]

\[c(t) = \cos \sqrt{3}t \quad s(t) = -i\sqrt{3} \sin \sqrt{3}t\]
FIG. 3. The quantum circuit for implementing the $4 \times 4$ unitary transformation $U$ on a two-qubit system.

FIG. 5. Practical experimental setup with consideration of both compensation of optical delay between different spatial modes and simplification.
FIG. 6. Photon probability distributions after eight unitary transformations. Red bars represent the experimental results. Blue borders represent the theoretical predictions. Errors are estimated via propagated Poissonian statistics.
PT-symmetric CTQWs on a Directed Graph
\[ H = \begin{bmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \]

Non-Hermitian
Possible solutions

**Szegedy quantum walk**
Expands the statespace, and embeds the non-unitary behaviour in a larger, unitary evolution operator.
- T. Loke et al. (2017)

**Quantum stochastic walk**
Introduces environmental decoherence and classical effects.
- Falloon et al. (2016)
- T. Loke et al. (2017)

**PT-symmetry**
Uses a similarity transform to map the non-unitary operator to a unitary one, whilst preserving defining characteristics.
- Salimi and Sorouri (2010).
PT-Symmetry


If a non-Hermitian matrix has PT symmetry,

\[ [\mathcal{PT}, H] = 0 \]

where

\[ \mathcal{P}: (x, y, z) \rightarrow (-x, -y, -z) \]
\[ \mathcal{T}: t \rightarrow -t \text{ and } \{\mathcal{T}, i\} = 0 \]

then it has **real eigenvalues** \( \lambda_j \in \mathbb{R} \):

\[ H|\phi_j\rangle = \lambda_j|\phi_j\rangle \text{ and } \mathcal{PT}|\phi_j\rangle = |\phi_j\rangle \]
Pseudo-Hermiticity

- Introduced by Mostafazadeh as a generalisation to Hermiticity

- A subset of generalised PT-symmetry: this is an extension of PT-symmetry, where \( \mathcal{P} \) is any arbitrary linear operator, and \( \mathcal{T} \) is any arbitrary anti-linear operator.

- Pseudo-Hermiticity is a necessary and sufficient condition for \( H \) to admit anti-linear symmetry
$H$ is pseudo-Hermitian if it satisfies any one of the following equivalent conditions:

- $\exists$ an Hermitian operator $\eta$ such that $H$ is similar to an Hermitian matrix $\tilde{H}$: $\tilde{H} = \eta H \eta^{-1}$

- $\exists$ an Hermitian operator $\eta$ such that $H$ is similar to its own Hermitian conjugate: $H^\dagger = \eta^2 H \eta^{-2}$

- $H$ has real eigenvalues and is diagonalisable

- $H$ has real eigenvalues and $n$ linearly-independent eigenvectors

Redefining the inner-product in the following form is sufficient to preserve the systems probability:

$$\langle \cdots | \cdots \rangle_\eta := \langle \cdots | \eta^2 | \cdots \rangle$$
Pseudo-Hermitian CTQW

\[ |\psi(t)\rangle = \tilde{U}(t)|\psi(0)\rangle = \eta e^{-iHt}\eta^{-1}|\psi(0)\rangle \]

\[ H = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
0 & 0 & 2 \\
\end{bmatrix} \]

\[ \eta = \frac{1}{6} \begin{bmatrix}
3 + 2\sqrt{2} & -3 + 2\sqrt{2} & \sqrt{2} \\
-3 + 2\sqrt{2} & 3 + 2\sqrt{2} & \sqrt{2} \\
\sqrt{2} & \sqrt{2} & 5\sqrt{2} \\
\end{bmatrix} \]

\[ \Rightarrow \quad \tilde{H} = \eta H \eta^{-1} = \frac{1}{9} \begin{bmatrix}
10 & 8 & -4 \\
-8 & 10 & -4 \\
-4 & -4 & 16 \\
\end{bmatrix} \]
<table>
<thead>
<tr>
<th>Vertex</th>
<th>PageRank</th>
<th>CTQW</th>
<th>η-CTQW</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.386364</td>
<td>0.339192</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.113636</td>
<td>0.013636</td>
</tr>
<tr>
<td>3</td>
<td>0.25</td>
<td>0.0160808</td>
<td>0.0160808</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>0.0160808</td>
<td>0.0160808</td>
</tr>
</tbody>
</table>

Quantum Centrality Ranking
Outlook

• Explore the complexity for a class of pseudo-Hermitian graphs

• Extend the framework to multiple distinguishable walkers, multiple bosons, and multiple fermions

• Apply the pseudo-Hermitian CTQW framework to models of photosynthesis and electron transport
Thank you