Quantum Walks & algorithms

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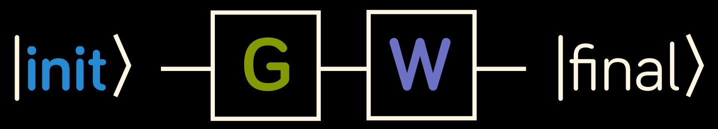
In part based on joint work with
Cătălin Dohotaru and Mojtaba Komeili

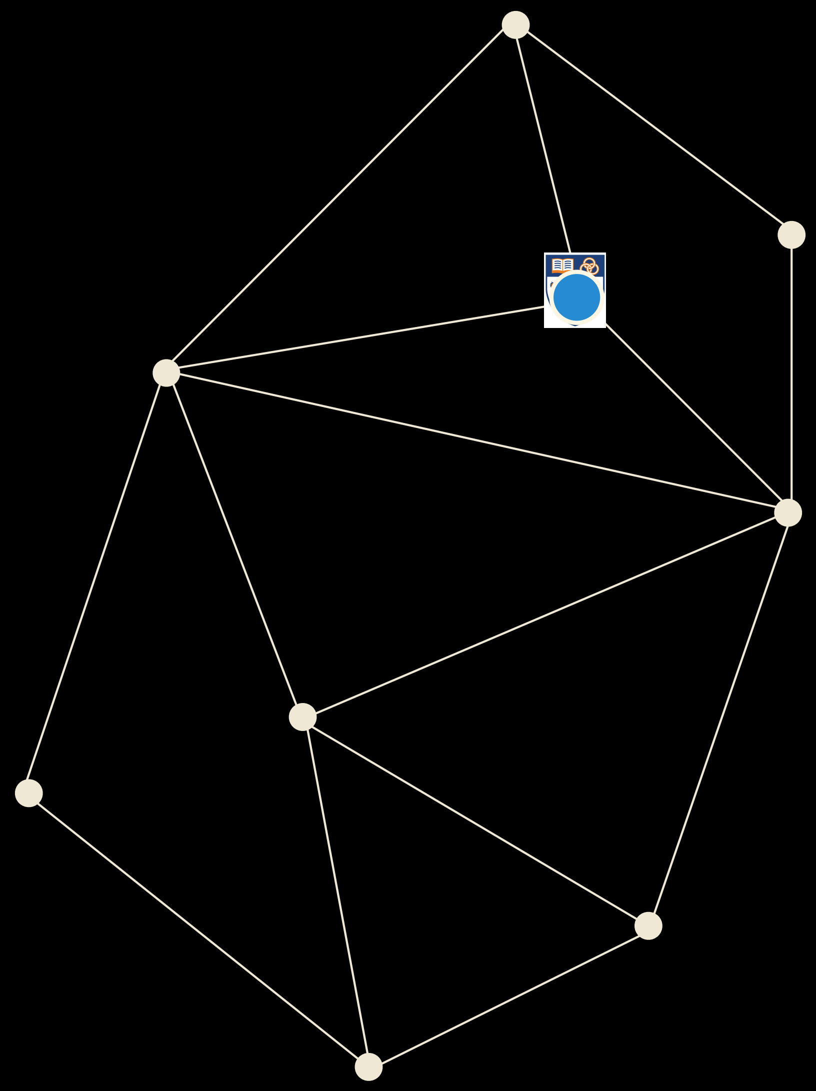
in arXiv:1612.08958,
QIP 2017, STACS 2017, ICALP 2017

Quantum algorithm

$$|\text{final}\rangle = (WG)^T |\text{init}\rangle$$



Random walk

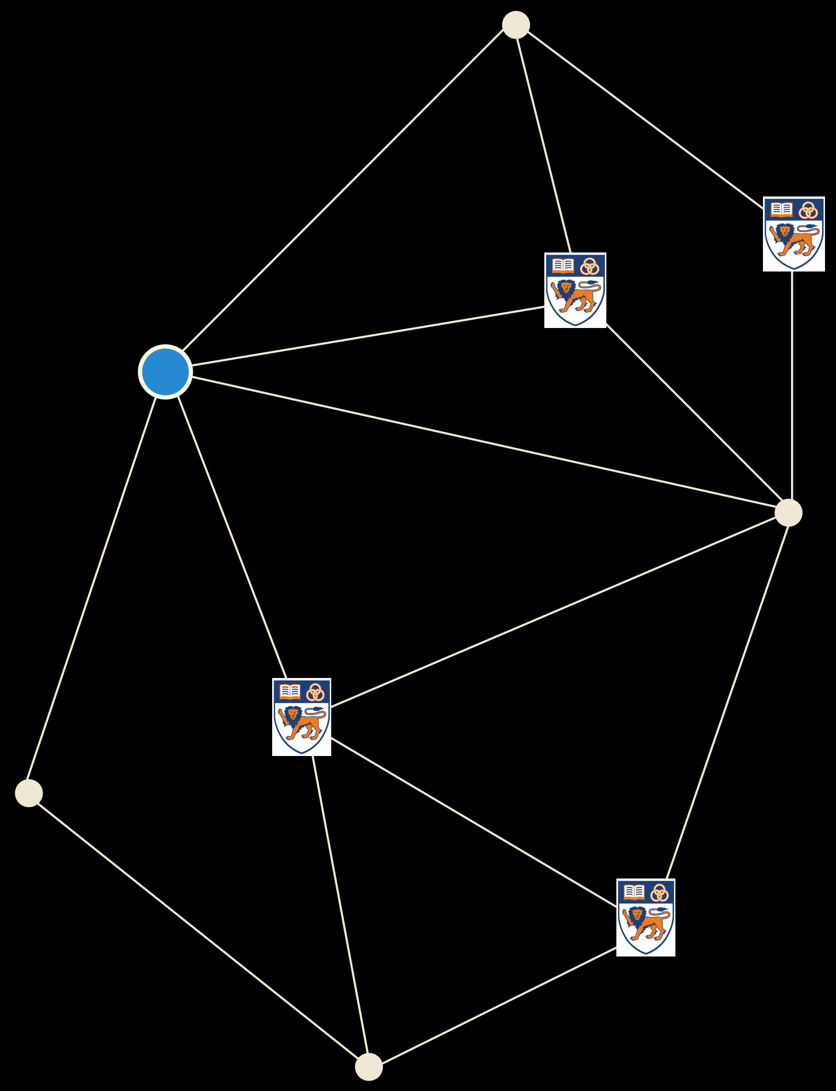


Setup walk at a random vertex
Repeat **T** times
- Check if vertex solution. If so, halt.
- Update by walking to a random neighbor

T = Hitting Time(P, AQIS)

Cost $=$ **S** + T (**C** + **U**)

Random walk



Setup walk at a random vertex
Repeat **T** times
- Check if vertex solution. If so, halt.
- Update by walking to a random neighbor

T = Hitting Time(P, **M**)

Cost = **S** + T (**C** + **U**)

Quantum Walk

Setup $|\mathbf{init}\rangle$

Repeat T_q times

- Check **G**
- Update **W**

$$|\mathbf{init}\rangle \; - \boxed{\mathbf{G}} - \boxed{\mathbf{W}} - \; |\mathrm{final}\rangle$$

$$\text{Cost} = \mathbf{S} + T_{\text{quantum}} (\mathbf{C} + \mathbf{U})$$

Want: $\mathbf{T_{quantum}} = \sqrt{\mathbf{T_{classical}}}$

Checking query (G)

Goal: Find a marked g

$$|x\rangle \mapsto \begin{cases} -|x\rangle & \text{if x is marked} \\ |x\rangle & \text{otherwise} \end{cases}$$

We can ask questions of the form:
"Is vertex x marked or not?"

$$\mathrm{Refl}(g) = \mathrm{Id} - 2|g\rangle\langle g|$$

$$\mathrm{Refl}(g) = \mathrm{Id} - 2 \sum_{g \text{ marked}} |g\rangle\langle g|$$

ε = Prob[random vertex is marked]

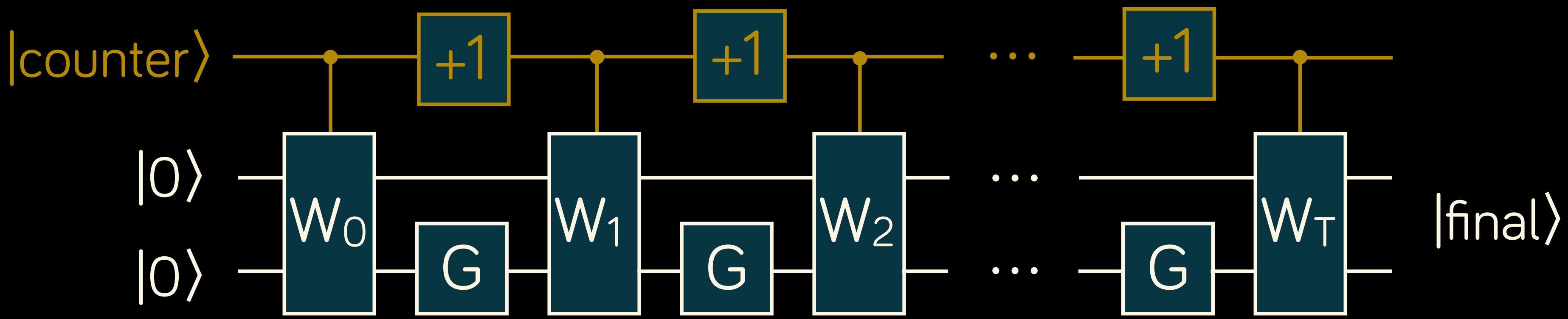
1 query to **G** ≡ 1 checking cost

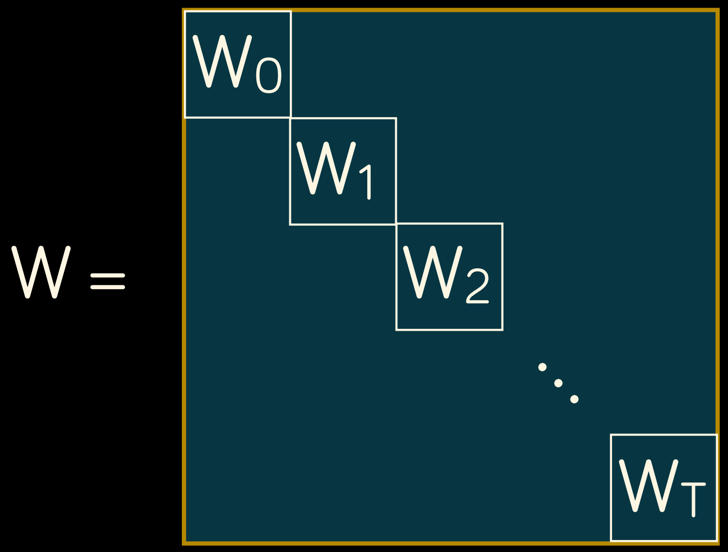
$$|x\rangle \mapsto (-1)^{[x \text{ marked}]}|x\rangle$$

$$|x\rangle|b\rangle \mapsto |x\rangle|b \oplus [x \text{ marked}]\rangle$$

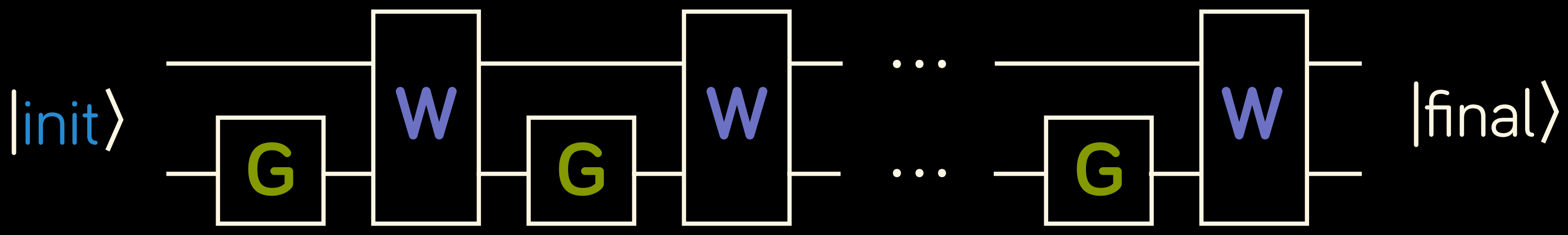
$$|x\rangle|b\rangle \mapsto (-1)^{b\cdot[x \text{ marked}]}|x\rangle|b\rangle$$

Universal quantum algorithm





Universal quantum algorithm



$$|\text{final}\rangle = (WG)^T |\text{init}\rangle$$

$$\text{Total Cost} = S + T\,(C + U)$$

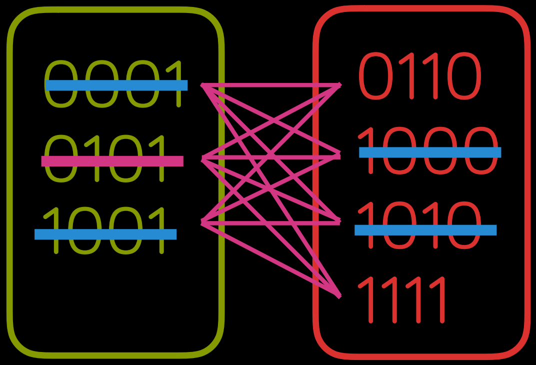
S = Setup cost
C = Checking cost
U = Update cost

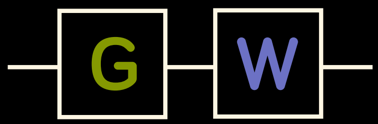
A universal construction of W

$X = X_1 X_2 X_3 X_4$

Query: $X_2 = 1$

Query: $X_3 = 1$





$|\text{final}\rangle = (WG)^T |\text{init}\rangle$

	0110	1000	1010	1111
0001	3	2	3	3
0101	2	3	4	2
1001	4	1	2	2

$\longrightarrow \quad |\Psi_x\rangle \quad \overset{|\Psi_y\rangle}{\boxed{\langle \Psi_x | \Psi_y \rangle}} \quad \longrightarrow \quad \Delta = \boxed{\langle \Psi_x | \Psi_{x'} \rangle}$

$$\#\text{Queries} \geq \max_\Gamma \frac{\|\Gamma\|}{\max_i \|\Gamma_i\|}$$

W = reflection about +1 eigenspace of Δ

Four algorithms

$|\text{final}\rangle = (\mathbf{WG})^{\mathbf{T}} |\mathbf{init}\rangle$

Goal: Find a marked g

	1-dim **G**	m-dim **G**
1-dim **W**		
Real unitary **W**		

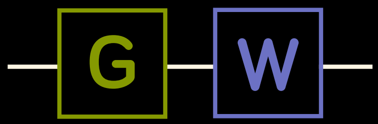
W = Refl(init) = $2|\text{init}\rangle\langle\text{init}| - \text{Id}$

W = real unitary

$\text{Refl}(g) = \text{Id} - 2|g\rangle\langle g|$

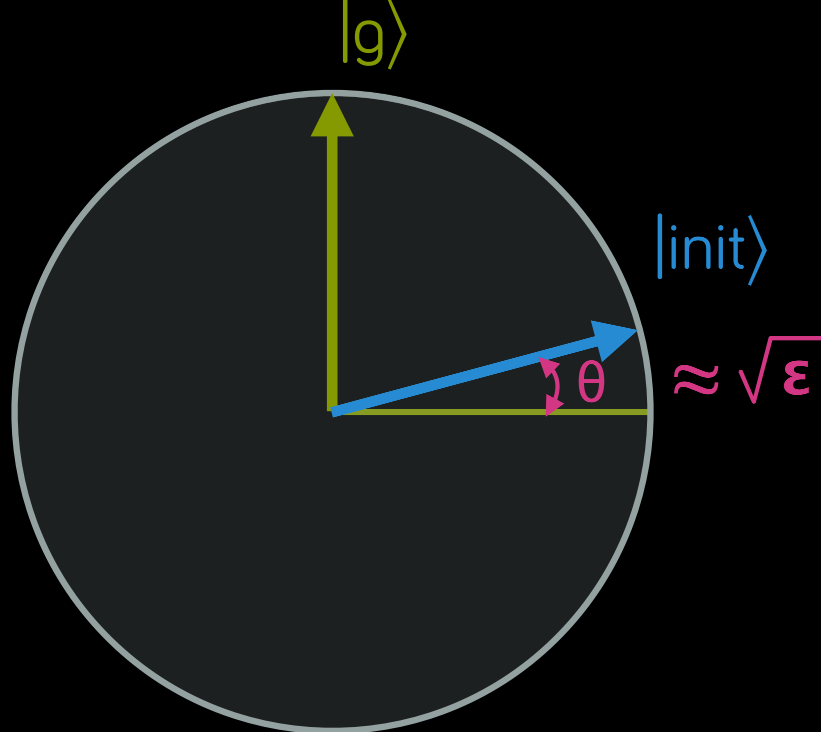
$\text{Refl}(g) = \text{Id} - 2 \sum_{g \text{ marked}} |g\rangle\langle g|$

Two one-dimensional reflections



Find a marked g

	1-dim **G**	m-dim **G**
1-dim **W**	now	
Real **W**		



$\mathbf{G} = \mathrm{Id} - 2|g\rangle\langle g|$

$\mathbf{W} = 2|init\rangle\langle init| - \mathrm{Id}$

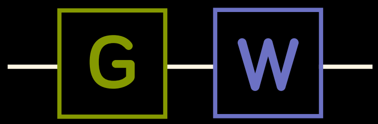
$\mathrm{Refl}(init) \circ \mathrm{Refl}(g) = \mathrm{Rot}(2\theta)$

$[\mathrm{Refl}(init) \circ \mathrm{Refl}(g)]^{\sqrt{1/\varepsilon}} \; |init\rangle \approx |g\rangle$

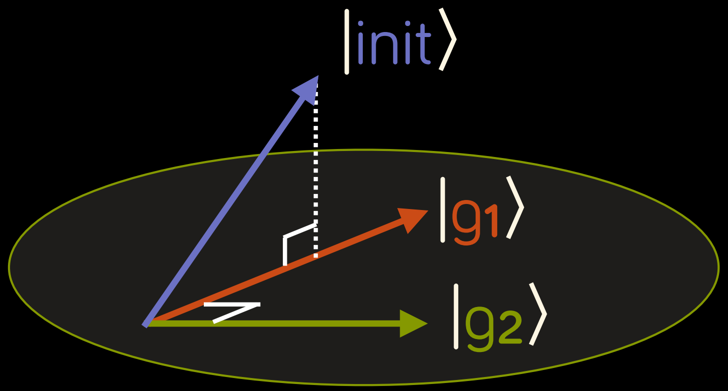
$\langle g|init\rangle = \sin(\theta) \approx \sqrt{\varepsilon}$

$$\boxed{\text{Cost(Rotate } |init\rangle \text{ by } 90^\circ) = \mathbf{S} + \sqrt{1/\varepsilon}\,(\mathbf{C} + \mathbf{U})}$$

Multiple solutions



Find a marked g



	1-dim **G**	m-dim **G**
1-dim **W**	2D rot	now
Real **W**		

$$\text{Refl}(g) = \text{Id} - 2 \sum_{g \text{ marked}} |g\rangle\langle g|$$

$$\mathbf{W} = 2|\text{init}\rangle\langle\text{init}| - \text{Id}$$

Let $|g_1\rangle = \sum_{g \text{ marked}} |g\rangle\langle g| \, |\text{init}\rangle$, normalized

Let $|g_2\rangle \perp |g_1\rangle$ So, $\mathbf{G} \, |g_2\rangle = - |g_2\rangle$

Then $|g_2\rangle \perp |\text{init}\rangle$ So, $\mathbf{W} \, |g_2\rangle = - |g_2\rangle$

$\mathbf{WG} \, |g_2\rangle = |g_2\rangle$

WG rotates in $\{ |\text{init}\rangle, |g_1\rangle \}$

General W with unique solution

—[G]—[W]— Find a marked g

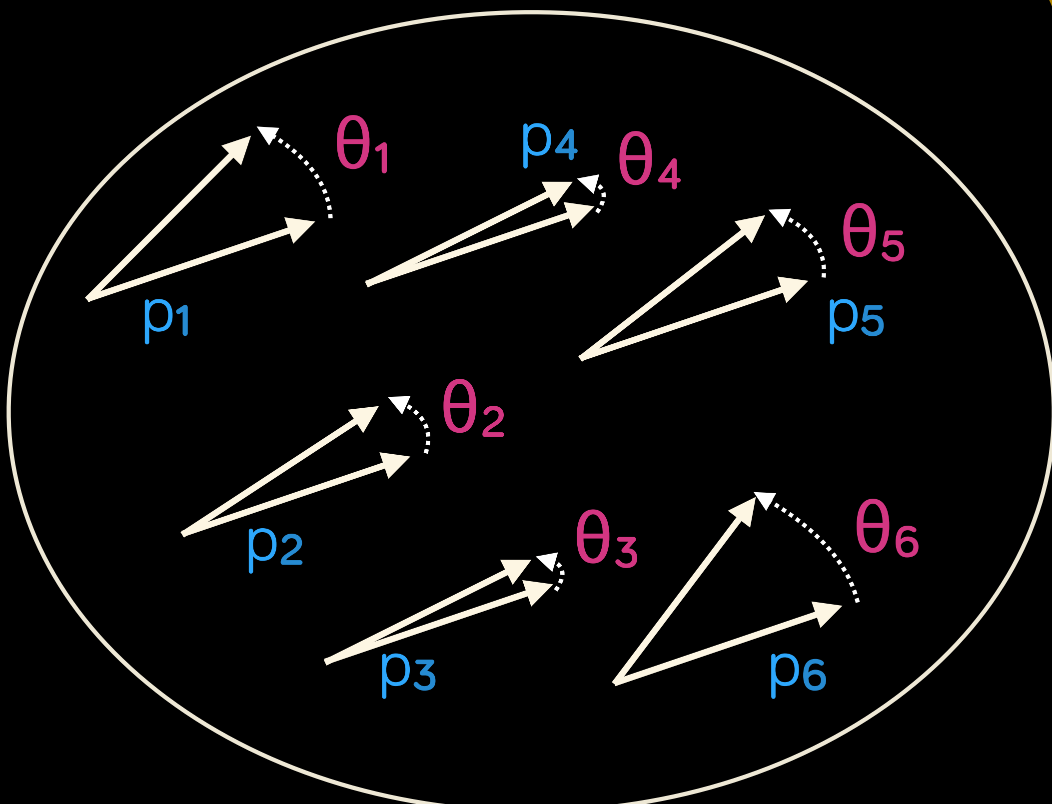
	1-dim **G**	m-dim **G**
1-dim **W**	2D rot	2D rot
Real **W**	now	

$\mathrm{Refl}(g) = \mathrm{Id} - 2|g\rangle\langle g|$

W real-valued

$W\,|\mathrm{init}\rangle = |\mathrm{init}\rangle$

$W\,|\mathrm{init}^{\perp}\rangle \neq |\mathrm{init}^{\perp}\rangle$



$$\sqrt{\frac{p_1}{\theta_1^2} + \frac{p_2}{\theta_2^2} + \frac{p_3}{\theta_3^2} + \frac{p_4}{\theta_4^2} + \frac{p_5}{\theta_5^2} + \frac{p_6}{\theta_6^2}}$$

$= \mathrm{Cost}(\text{Rotate } |\mathrm{init}\rangle \text{ by } 90°)$

$= \mathrm{Cost}(\text{Phase estimation})$

$= \mathrm{QHT}(W \circ \mathrm{Refl}(g), |\mathrm{init}\rangle)$

$= \sqrt{\mathrm{HT}} = \sqrt{\text{Hitting Time}}$

$$[W \circ \mathrm{Refl}(g)]^{\sqrt{\mathrm{HT}}}\ |\mathrm{init}\rangle \begin{cases} \neq |\mathrm{init}\rangle \\ \neq |g\rangle \end{cases}$$

General W with unique solution

W real-valued

$W\,|\text{init}\rangle = |\text{init}\rangle$

$W\,|\text{init}^\perp\rangle \neq |\text{init}^\perp\rangle$

$$W = \begin{bmatrix} +1 & & & & \\ & \varphi_1 & & & \\ & & \varphi_2 & & \\ & & & \varphi_3 & \\ & & & & \varphi_4 \end{bmatrix} \qquad \mathrm{Refl(init)} = \begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}$$

$$\text{Cost} \approx \mathbf{S} + \sqrt{HT}\,(\mathbf{C} + \mathbf{U}) \qquad \text{Cost} \approx \mathbf{S} + \sqrt{1/\varepsilon}\,(\mathbf{C} + \mathbf{U})$$

$$W^{\mathbf{E}} = \begin{bmatrix} +1 & & & & \\ & E\varphi_1 & & & \\ & & E\varphi_2 & & \\ & & & E\varphi_3 & \\ & & & & E\varphi_4 \end{bmatrix} \not\approx \begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix} = \mathrm{Refl(init)}$$

$$[W^{\mathbf{E}} \circ \mathrm{Refl}(g)]^{\sqrt{1/\varepsilon}}\,|\text{init}\rangle \not\approx |g\rangle$$

$$\text{Cost} \approx \mathbf{S} + \sqrt{1/\varepsilon}\,\mathbf{C} + \sqrt{1/\varepsilon}\,E\,\mathbf{U}$$

General W with unique solution

Thm: $$\frac{W^{E}|g\rangle \approx |g^{\perp}\rangle \qquad W^{E}|\text{init}\rangle = |\text{init}\rangle}{W^{E} = \widetilde{\text{Refl}}(\text{init})}$$

for $E = \sqrt{\varepsilon * HT}$

by using phase estimation

Cor: $$[\widetilde{\text{Refl}}(\text{init}) \circ \text{Refl}(g)]^{\sqrt{1/\varepsilon}}|\text{init}\rangle \not\approx |g\rangle$$

$$[W^{E} \circ \text{Refl}(g)]^{\sqrt{1/\varepsilon}}|\text{init}\rangle \not\approx |g\rangle$$

$$\text{Cost} \approx \mathbf{S} + \sqrt{1/\varepsilon}\ \mathbf{C} + \sqrt{1/\varepsilon}\ E\ \mathbf{U}$$

General W with unique solution

$$[\widetilde{\mathrm{Refl}}(\mathrm{init}) \circ \mathrm{Refl}(g)]^{\sqrt{1/\varepsilon}}\,|\mathrm{init}\rangle \;\cancel{\approx}\; |g\rangle$$

W G W G W G W G W G W G W G W G W G W G W G W G W G

(W G) W (G W) G (W G)) W ((G W) G (W G) W (G W)) G ((W G) W (G W) G (W G))

Recursive form: $U\,U^{-1}\,U$

Do recursively: $[\widetilde{\mathrm{Refl}}(\mathrm{init}) \circ \mathrm{Refl}(g)]^{\sqrt{1/\varepsilon}}\,|\mathrm{init}\rangle \approx |g\rangle$

Cost $\approx$ **S** + $\sqrt{1/\varepsilon}$ **C** + $\sqrt{HT}$ **U**

Overview

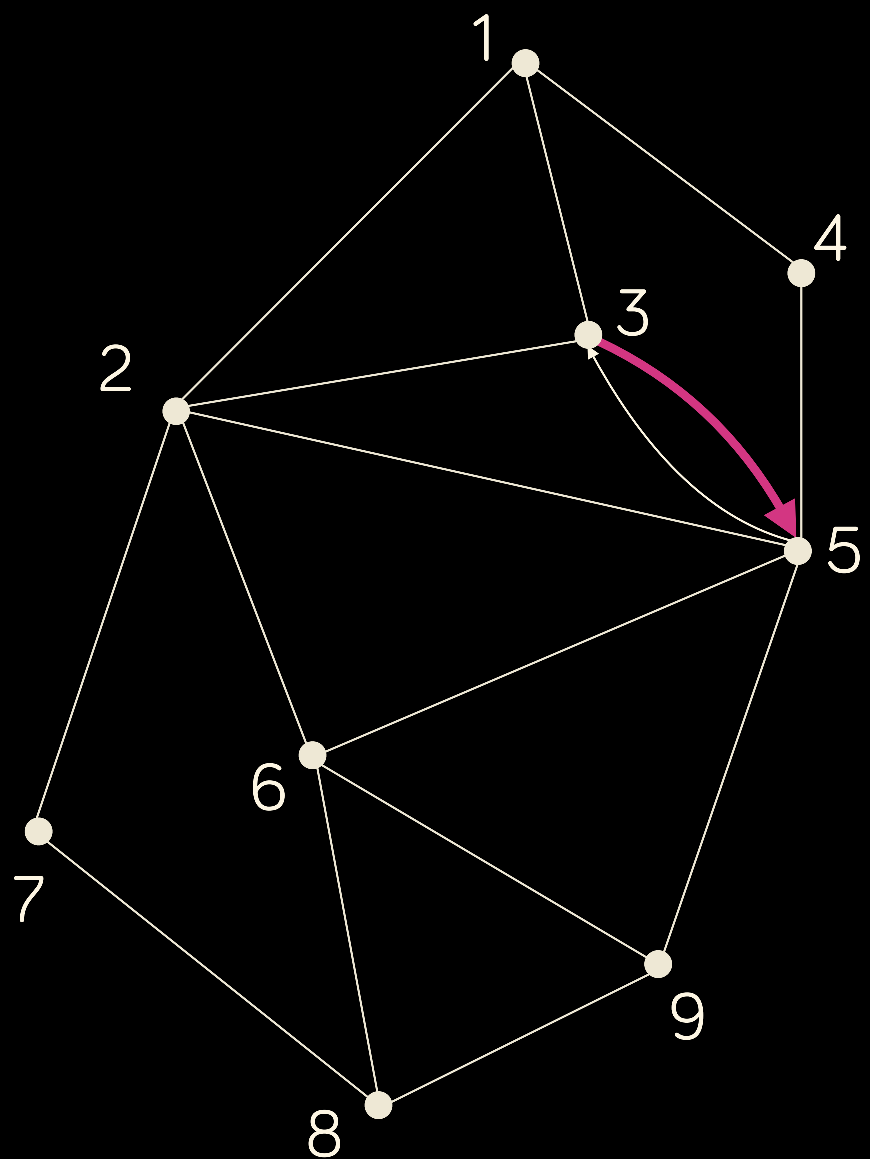
$$|\mathrm{init}\rangle \; -\boxed{G}-\boxed{W}- \; |\mathrm{final}\rangle \approx |g\rangle$$

	1-dim **G**	m-dim **G**
1-dim **W**	2D rot	2D rot
Real **W**	Rec. Ampl. Ampl	Mostly open

$$\mathrm{Cost}(\text{1-dim } W) \approx S + \sqrt{1/\varepsilon}\,(C + U)$$

$$\mathrm{Cost}(\text{real } W) \approx S + \sqrt{1/\varepsilon}\; C + \sqrt{HT}\; U$$

Random walk

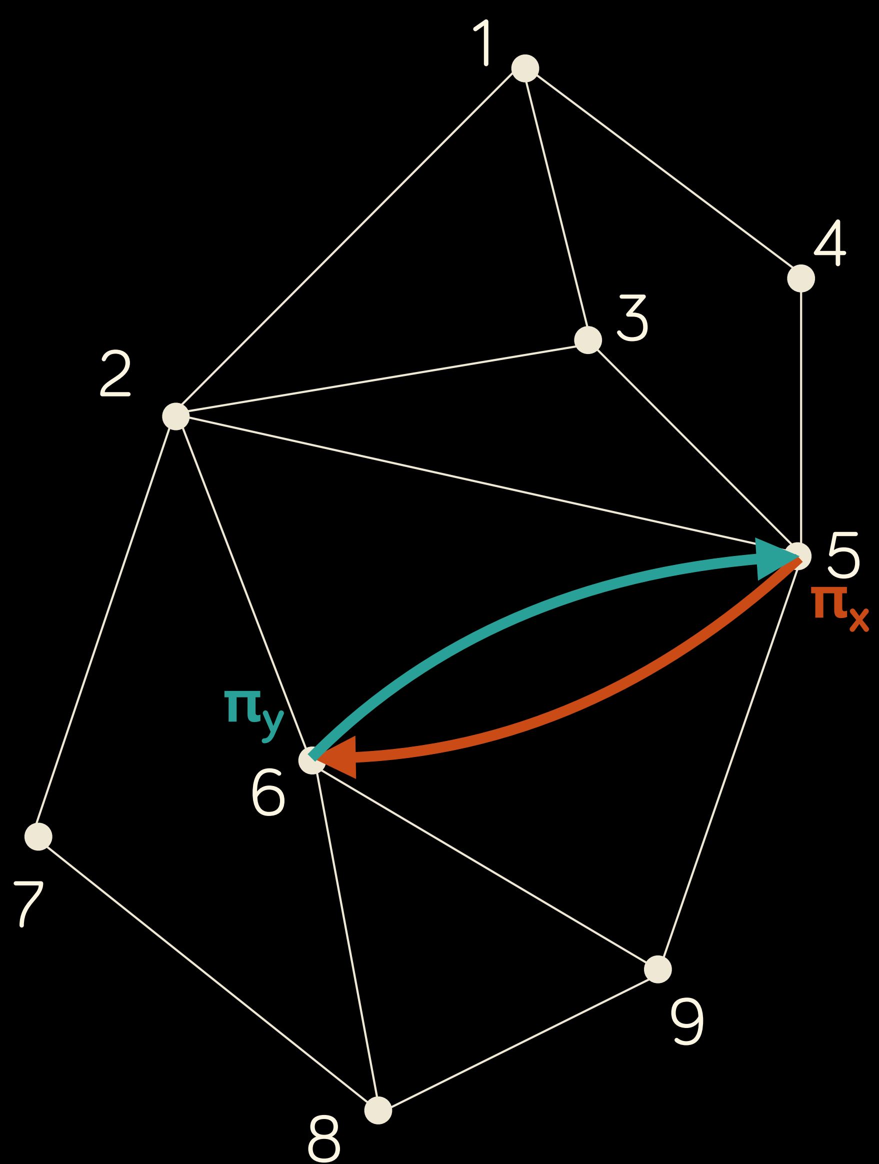


Probability to transit from 3 to 5 = $p_{5\leftarrow 3}$

$P =$

	1/5	1/3	1/2					
1/3		1/3		1/5	1/4	1/2		
1/3	1/5			1/5				
1/3				1/5				
	1/5	1/3	1/2		1/4			1/3
	1/5			1/5			1/3	1/3
	1/5						1/3	
					1/4	1/2		1/3
				1/5	1/4		1/3	

Random walk



Columns sum to 1

Connected

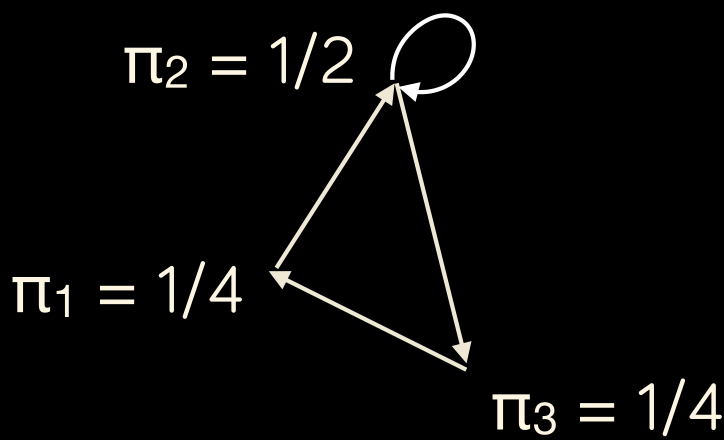
Stationary distr.: $P\vec{\pi} = \vec{\pi}$

Reversible: $p_{y \leftarrow x}\,\pi_x = p_{x \leftarrow y}\,\pi_y$

P =

.1	.2	.1		.2	.2		.1	
.3		.2	.1		.3		.2	
.2			.4					.1
.4	.3			.3	.3	.1		.6
	.1	.6			.3			.1
	.2		.2	.4				.1
	.2		.3		.2		.4	
		.1				.9	.2	
				.1			.1	.1

A non-reversible graph



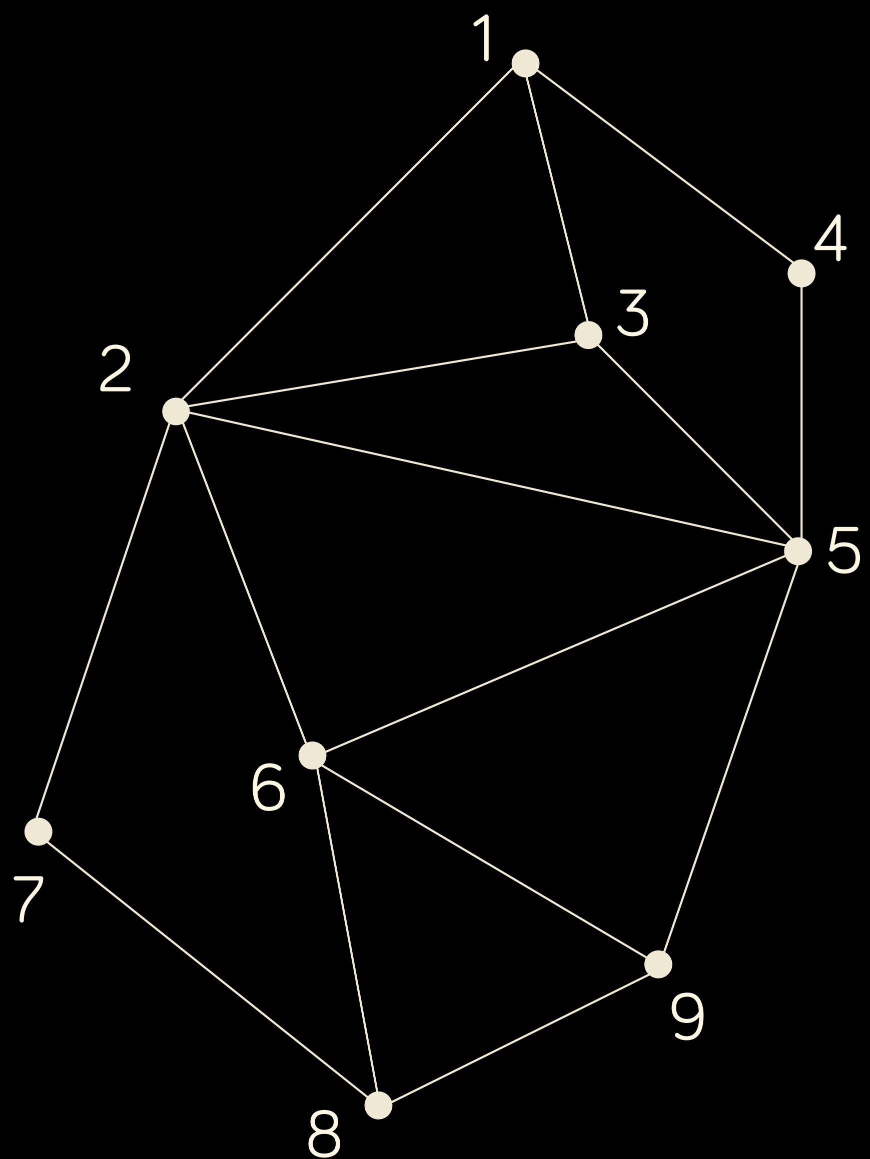
Columns sum to 1

Connected

Stationary distr.: $P\vec{\pi} = \vec{\pi}$

Reversible: $p_{y \leftarrow x}\, \pi_x \neq p_{x \leftarrow y}\, \pi_y$

Random walk



Columns sum to 1

Connected

Stationary distr.: $P\vec{\pi} = \vec{\pi}$

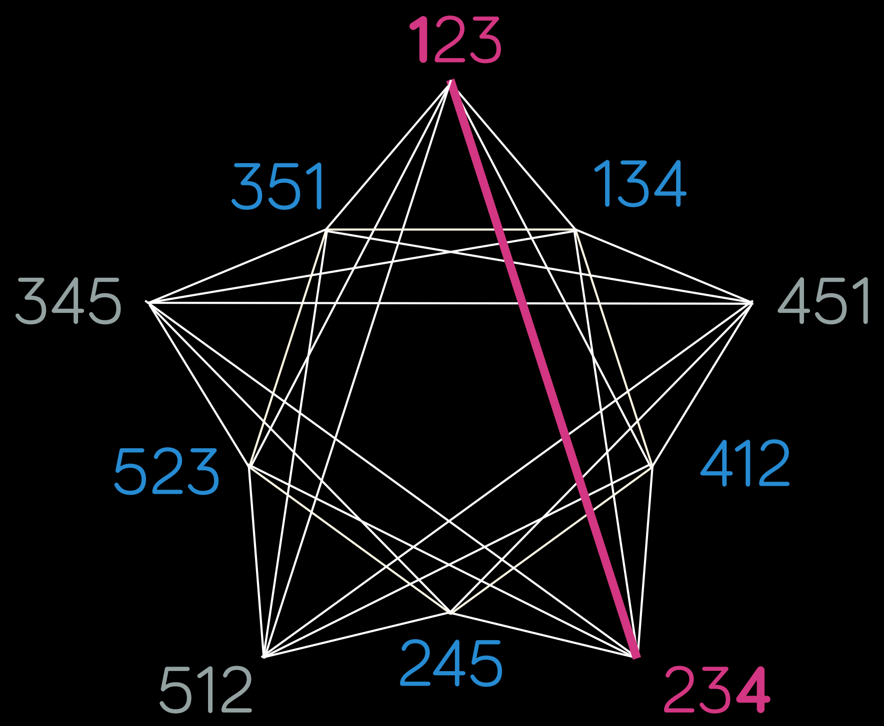
Reversible: $p_{y\leftarrow x}\,\pi_x = p_{x\leftarrow y}\,\pi_y$

ε = Prob [random vertex is marked when picked from $\vec{\pi}$]

P =

	1/5	1/3	1/2					
1/3		1/3		1/5	1/4	1/2		
1/3	1/5			1/5				
1/3				1/5				
	1/5	1/3	1/2		1/4			1/3
	1/5			1/5			1/3	1/3
	1/5						1/3	
					1/4	1/2		1/3
				1/5	1/4		1/3	

Hitting time



Unique marked vertex

	Hitting Time	ε
Complete graph	$1/\varepsilon$	$1/N$
2D Grid	$1/\varepsilon \cdot \log(N)$	$1/N$
3D Grid	$1/\varepsilon$	$1/N$
[N choose k] (Johnson graph)	$1/\varepsilon \cdot k$	$(k/N)^2$

$k = \text{poly}(N)$ for our applications

Vertex = Subset of size **k** out of $[N] = \{1, 2, \ldots, N\}$

Edge = Two subsets differ in one element

Random walk

$$P = \begin{array}{|c|c|c|c|c|} \hline .1 & .2 & .2 & .3 & .3 \\ \hline .4 & .3 & .1 & 0 & 0 \\ \hline 0 & .1 & .1 & .7 & .4 \\ \hline 0 & .2 & .4 & 0 & .2 \\ \hline .5 & .2 & .2 & 0 & .1 \\ \hline \end{array}$$

Columns sum to 1

Connected

Stationary distr.: $P\vec{\pi} = \vec{\pi}$

Reversible: $p_{y\leftarrow x}\,\pi_x = p_{x\leftarrow y}\,\pi_y$

$$\sqrt{\pi} = \begin{pmatrix} \sqrt{\pi_1} & & & 0 \\ & \sqrt{\pi_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\pi_N} \end{pmatrix}$$

$$D(P) = \sqrt{P^T \circ P} = \sqrt{1/\pi}\; P \sqrt{\pi}$$

$$D(P)_{y\leftarrow x} = \sqrt{p_{y\leftarrow x}\, p_{x\leftarrow y}} = \langle y|p_x\rangle\langle x|p_y\rangle = \langle y, p_y | p_x, x\rangle = \langle y, p_y | S | x, p_x\rangle = \langle y | T^\dagger S T | x\rangle$$

$$(P^3)_{y\leftarrow x} = ((\sqrt{\pi}\, D(P)\sqrt{1/\pi}\,)^3)_{y\leftarrow x} = \langle y|\sqrt{\pi}\; T^\dagger S T\; T^\dagger S T\; T^\dagger S T\; \sqrt{1/\pi}\,|x\rangle$$

$$= \langle y|\sqrt{\pi}\; T^\dagger S T\; T^\dagger S T\; T^\dagger S T\; \sqrt{1/\pi}\,|x\rangle = \sqrt{\pi_y}\,\langle y, p_y | (S\,T\,T^\dagger)^3 | x, p_x\rangle \sqrt{1/\pi_x}$$

$|p_x\rangle = \sum \sqrt{p_{y\leftarrow x}}\; |y\rangle$

$S = \text{Swap} = \sum |x, y\rangle\langle y, x|$

$T|x\rangle = |x, p_x\rangle$

$T = \sum |x, p_x\rangle\langle x|$

$T T^\dagger = \sum |x, p_x\rangle\langle x, p_x|$

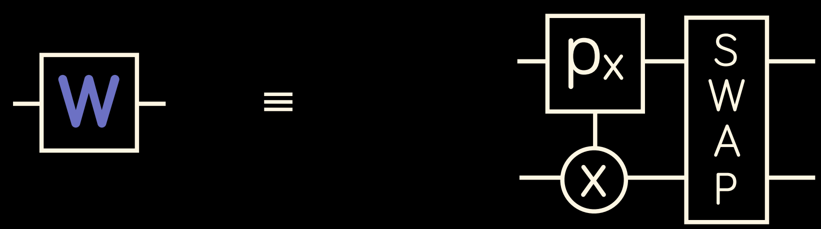
$T^\dagger T = \text{Id}$

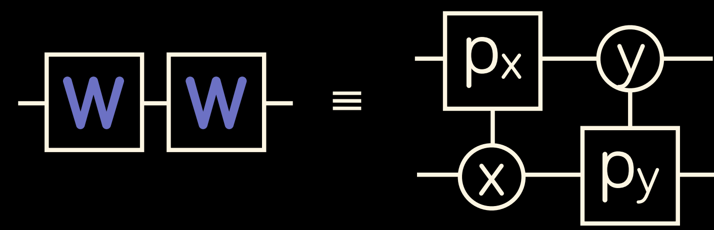
Quantum walk

Stationary distr.: $P\vec{\pi} = \vec{\pi}$ $\qquad D(P) = \sqrt{P^T \circ P} = \sqrt{1/\pi}\; P \sqrt{\pi}$

Reversible: $p_{y\leftarrow x}\,\pi_x = p_{x\leftarrow y}\,\pi_y$ $\qquad D(P)_{y\leftarrow x} = \langle y, p_y | \mathbf{S}\,\mathbf{T}\mathbf{T}^\dagger | x, p_x\rangle = \langle y | \mathbf{T}^\dagger \mathbf{S}\,\mathbf{T} | x\rangle$

$\mathbf{W}(P) = \mathbf{S}(2\mathbf{T}\mathbf{T}^\dagger - \mathrm{Id})$





$|p_x\rangle = \sum \sqrt{p_{y\leftarrow x}}\; |y\rangle$ $\qquad T\,|x\rangle = |x, p_x\rangle$ $\qquad T\,T^\dagger = \sum |x, p_x\rangle\langle x, p_x|$

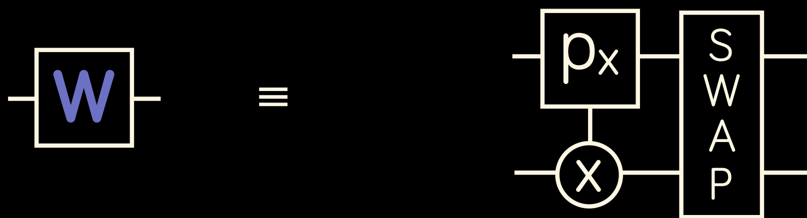
$S = \mathrm{Swap} = \sum |x, y\rangle\langle y, x|$ $\qquad T = \sum |x, p_x\rangle\langle x|$ $\qquad T^\dagger T = \mathrm{Id}$

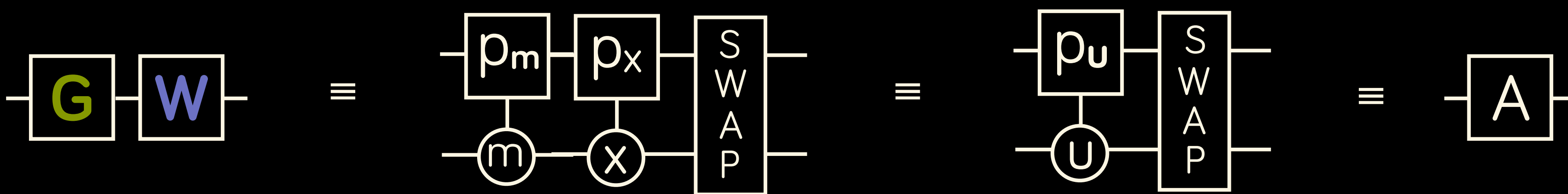
Quantum walk

Stationary distr.: $P\,\vec{\pi} = \vec{\pi}$ $\qquad D(P) = \sqrt{P^T \circ P} = \sqrt{1/\pi}\; P\, \sqrt{\pi}$

Reversible: $p_{y \leftarrow x}\, \pi_x = p_{x \leftarrow y}\, \pi_y$ $\qquad D(P)_{y \leftarrow x} = \langle y, p_y | \mathbf{S}\, \mathbf{T}\mathbf{T}^\dagger | x, p_x \rangle = \langle y | \mathbf{T}^\dagger \mathbf{S}\, \mathbf{T} | x \rangle$

$W(P) = S(2TT^\dagger - \mathrm{Id})$ $\qquad |\mathrm{final}\rangle = (WG)^T |\mathrm{init}\rangle$ $\qquad A = WG$





$|p_x\rangle = \sum \sqrt{p_{y \leftarrow x}}\; |y\rangle$ $\qquad T\,|x\rangle = |x, p_x\rangle$ $\qquad TT^\dagger = \sum |x, p_x\rangle\langle x, p_x|$

$S = \mathrm{Swap} = \sum |x, y\rangle\langle y, x|$ $\qquad T = \sum |x, p_x\rangle\langle x|$ $\qquad T^\dagger T = \mathrm{Id}$

Quantum walk

Stationary distr.: $P\vec{\pi} = \vec{\pi}$

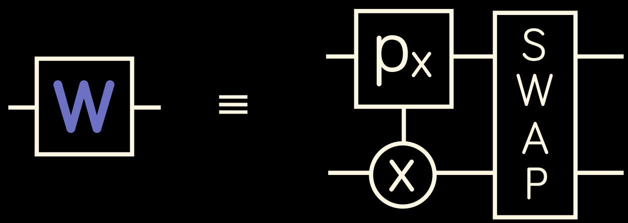
Reversible: $p_{y \leftarrow x}\,\pi_x = p_{x \leftarrow y}\,\pi_y$

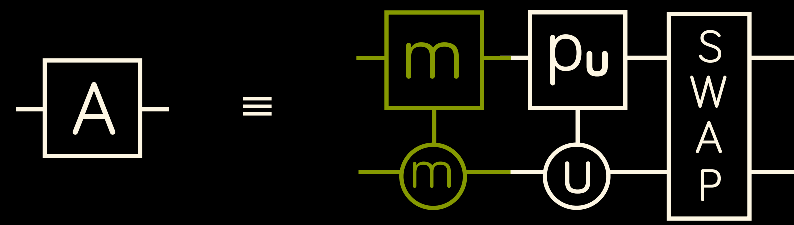
$D(P') = \sqrt{P'^{T} \circ P'} = \sqrt{1/\pi}\; P'\, \sqrt{\pi}$

$D(P')_{y \leftarrow x} = \langle y, p_y \,|\, S\,TT^{\dagger} \,|\, x, p_x \rangle = \langle y \,|\, T^{\dagger} S\, T \,|\, x \rangle$

$W(P) = S(2TT^{\dagger} - \mathrm{Id})$

$A = W(P') = S(2TT^{\dagger} - \mathrm{Id})$ $\quad A = WG$





$$P = \begin{bmatrix} .1 & .2 & .2 & .3 & .3 \\ .4 & .3 & .1 & 0 & 0 \\ 0 & .1 & .1 & .7 & .4 \\ 0 & .2 & .4 & 0 & .2 \\ .5 & .2 & .2 & 0 & .1 \end{bmatrix}$$

$$P' = \begin{bmatrix} .1 & .2 & .2 & 0 & 0 \\ .4 & .3 & .1 & 0 & 0 \\ 0 & .1 & .1 & 0 & 0 \\ 0 & .2 & .4 & 1 & 0 \\ .5 & .2 & .2 & 0 & 1 \end{bmatrix}$$

Marked = {4, 5}

$|p_x\rangle = \sum \sqrt{p_{y \leftarrow x}}\; |y\rangle$

$S = \mathrm{Swap} = \sum |x, y\rangle\langle y, x|$

$T\,|x\rangle = |x, p_x\rangle$

$T = \sum |x, p_x\rangle\langle x|$

$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$

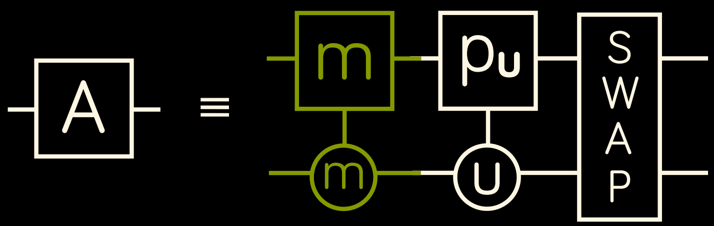
$TT^{\dagger} = \sum |x, p_x\rangle\langle x, p_x|$

$T^{\dagger}T = \mathrm{Id}$

Quantum walk

$$D(P')_{y \leftarrow x} = \langle y | T^\dagger S T | x \rangle$$
$$= \langle y, p_y | S T T^\dagger | x, p_x \rangle$$

$$A = W(P') = S(2TT^\dagger - \mathrm{Id})$$



$$D(P') = \begin{pmatrix} .1 & .3 & .1 & 0 & 0 \\ .3 & .2 & .1 & 0 & 0 \\ .1 & .1 & .1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$$

Spectra of D(P′) and W(P′):

Pick $T^\dagger S T \vec{\lambda}_D = \lambda \vec{\lambda}_D \qquad (\lambda < 1)$

Consider $|\lambda\rangle = T \vec{\lambda}_D$

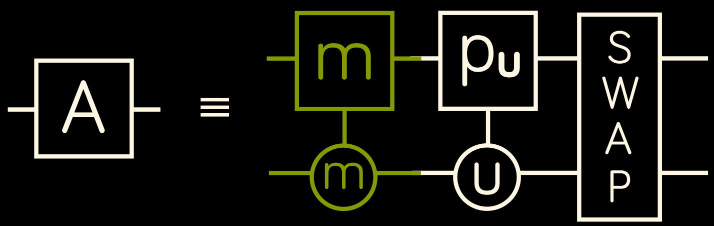
$$W(P')|\lambda\rangle$$
$$= S(2TT^\dagger - \mathrm{Id})|\lambda\rangle$$
$$= 2\, S T T^\dagger |\lambda\rangle - S|\lambda\rangle$$
$$= 2\, S T T^\dagger T \vec{\lambda}_D - S|\lambda\rangle$$
$$= 2\, S T \vec{\lambda}_D - S|\lambda\rangle$$
$$= 2\, S |\lambda\rangle - S|\lambda\rangle$$
$$= S|\lambda\rangle$$

$$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$$

Quantum walk

$D(P')_{y \leftarrow x} = \langle y | T^\dagger S T | x \rangle$
$\quad = \langle y, p_y | S T T^\dagger | x, p_x \rangle$

$A = W(P') = S(2TT^\dagger - \mathrm{Id})$



$D(P') =$

.1	.3	.1	0	0
.3	.2	.1	0	0
.1	.1	.1	0	0
0	0	0	1	0
0	0	0	0	1

$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$

Spectra of D(P′) and W(P′):

Pick $T^\dagger S T \vec{\lambda_D} = \lambda \vec{\lambda_D} \qquad (\lambda < 1)$

Consider $|\lambda\rangle = T \vec{\lambda_D}$

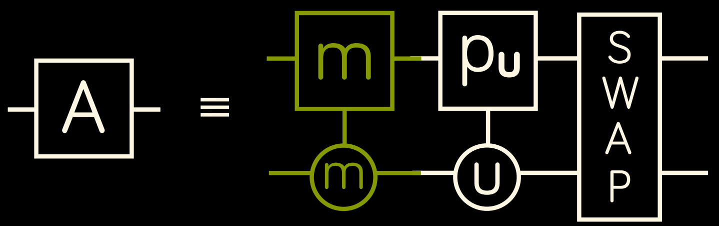
$W(P')|\lambda\rangle = S|\lambda\rangle$

$W(P')\, S|\lambda\rangle$
$= S(2TT^\dagger - \mathrm{Id})\, S\, |\lambda\rangle$
$= 2\, S T T^\dagger S|\lambda\rangle - |\lambda\rangle$
$= 2\, S T T^\dagger S T \vec{\lambda_D} - |\lambda\rangle$
$= 2\, \lambda\, S T \vec{\lambda_D} - |\lambda\rangle$
$= 2\, \lambda\, S|\lambda\rangle - |\lambda\rangle$

Quantum walk

$$D(P')_{y \leftarrow x} = \langle y | T^\dagger S T | x \rangle$$
$$= \langle y, p_y | S T T^\dagger | x, p_x \rangle$$

$$A = W(P') = S(2TT^\dagger - \mathrm{Id})$$



$$D(P') = \begin{pmatrix} .1 & .3 & .1 & 0 & 0 \\ .3 & .2 & .1 & 0 & 0 \\ .1 & .1 & .1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$$

Spectra of D(P′) and W(P′):

Pick $T^\dagger S T \vec{\lambda}_D = \lambda \vec{\lambda}_D \qquad (\lambda < 1)$

Consider $|\lambda\rangle = T \vec{\lambda}_D$

$$W(P')|\lambda\rangle = S|\lambda\rangle$$
$$W(P')\, S|\lambda\rangle = 2\lambda\, S|\lambda\rangle - |\lambda\rangle$$

W(P′) rotates 2D-subspace

$$\{ |\lambda\rangle, S|\lambda\rangle \}$$

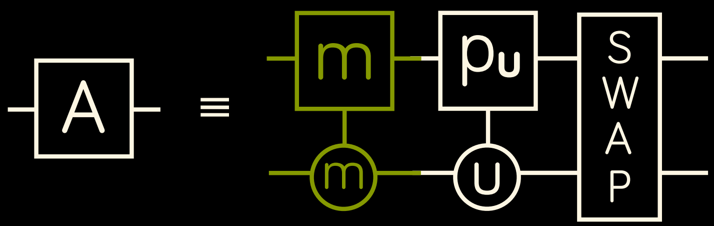
by angle

$$\langle\lambda|S|\lambda\rangle = \lambda = \cos(\Theta)$$

Quantum walk

$$D(P')_{y \leftarrow x} = \langle y | T^{\dagger} S\, T | x \rangle$$
$$= \langle y, p_y | S\, T T^{\dagger} | x, p_x \rangle$$

$$A = W(P') = S(2TT^{\dagger} - \mathrm{Id})$$



$$D(P') = \begin{pmatrix} .1 & .3 & .1 & 0 & 0 \\ .3 & .2 & .1 & 0 & 0 \\ .1 & .1 & .1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$$

Hitting Time of D(P′) and W(P′):

Expected #steps to reach a marked vertex, starting from the stationary distr. $\vec{\pi}$

$$\begin{array}{cc} D(P') & \sqrt{\vec{\pi}} \\ \begin{pmatrix} \ddots & & & 0 & 0 \\ & \vec{\lambda}_D & & 0 & 0 \\ & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$P' = \sqrt{\pi}\; D(P')\; \sqrt{1/\pi}$$

$$\sqrt{\vec{\pi}} = b_1 \vec{\lambda}_{D1} + b_2 \vec{\lambda}_{D2} + b_3 \vec{\lambda}_{D3}$$

$$\boxed{HT = \sum_j \frac{b_j^2}{1 - \lambda_{D\,j}}}$$

Quantum walk

$$QHT = \sqrt{\sum \frac{\text{Probability}}{\text{phase}^2}}$$

Hitting Time of $D(P')$ and $W(P')$:

Expected #steps to reach a marked vertex, starting from the stationary distr. $\vec{\pi}$

$W(P')$ rotates $|\lambda_1\rangle$ by angle $\theta_1 \approx \sqrt{1 - \lambda_1}$

since $\lambda_1 = \cos(\theta_1)$

$$|init\rangle = b_1|\lambda_1\rangle + b_2|\lambda_2\rangle + b_3|\lambda_3\rangle$$

$$QHT = \sqrt{\sum_j \frac{b_j^2}{1 - \lambda_{D\,j}}}$$

Quantum walk

Stationary distr.: $P\vec{\pi} = \vec{\pi}$

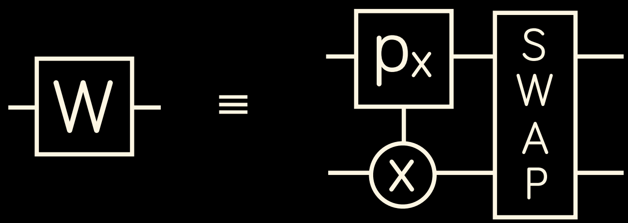
Reversible: $p_{y \leftarrow x}\,\pi_x = p_{x \leftarrow y}\,\pi_y$

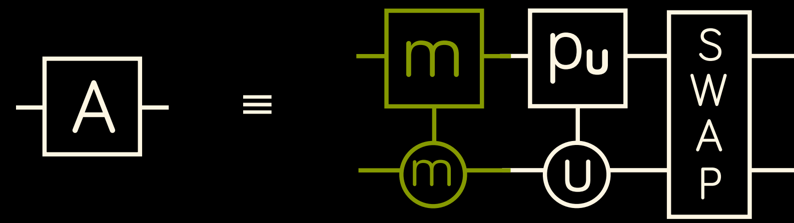
$D(P') = \sqrt{P'^{T} \circ P'} = \sqrt{1/\pi}\; P' \sqrt{\pi}$

$D(P')_{y \leftarrow x} = \langle y, p_y | S\,TT^{\dagger} | x, p_x \rangle = \langle y | T^{\dagger} S\, T | x \rangle$

$W(P) = S(2TT^{\dagger} - \mathrm{Id})$

$A = W(P') = S(2TT^{\dagger} - \mathrm{Id})$ $\quad A = WG$





$$P = \begin{bmatrix} .1 & .2 & .2 & .3 & .3 \\ .4 & .3 & .1 & 0 & 0 \\ 0 & .1 & .1 & .7 & .4 \\ 0 & .2 & .4 & 0 & .2 \\ .5 & .2 & .2 & 0 & .1 \end{bmatrix}$$

$$P' = \begin{bmatrix} .1 & .2 & .2 & 0 & 0 \\ .4 & .3 & .1 & 0 & 0 \\ 0 & .1 & .1 & 0 & 0 \\ 0 & .2 & .4 & 1 & 0 \\ .5 & .2 & .2 & 0 & 1 \end{bmatrix}$$

Marked = {4, 5}

$|p_x\rangle = \sum \sqrt{p_{y \leftarrow x}}\; |y\rangle$

$S = \mathrm{Swap} = \sum |x, y\rangle\langle y, x|$

$T|x\rangle = |x, p_x\rangle$

$T = \sum |x, p_x\rangle\langle x|$

$T = \sum |u, p_u\rangle\langle u| + \sum |m, m\rangle\langle m|$

$TT^{\dagger} = \sum |x, p_x\rangle\langle x, p_x|$

$T^{\dagger}T = \mathrm{Id}$

Quantum walk

Stationary distr.: $P\vec{\pi} = \vec{\pi}$ $\quad D(P) = \sqrt{P^T \circ P} = \sqrt{1/\pi}\ P\ \sqrt{\pi}$

Reversible: $p_{y\leftarrow x}\,\pi_x = p_{x\leftarrow y}\,\pi_y$ $\quad D(P)_{y\leftarrow x} = \langle y, p_y | \mathbf{S}\,\mathbf{T}\mathbf{T}^\dagger | x, p_x\rangle = \langle y | \mathbf{T}^\dagger \mathbf{S}\,\mathbf{T} | x\rangle$

$W(P) = \mathbf{S}(2\mathbf{T}\mathbf{T}^\dagger - \mathrm{Id})$

1. $W(P)\,|\mathrm{init}\rangle = |\mathrm{init}\rangle$ for $|\mathrm{init}\rangle = \mathbf{T}\sqrt{1/\pi}\,\vec{\pi} = \sum \sqrt{\pi_x}\,|x, p_x\rangle$

   $\mathbf{S}\,|\mathrm{init}\rangle = |\mathrm{init}\rangle$

2. $W(P)$ acts on $\{|x, p_x\rangle, \mathbf{S}|x, p_x\rangle\}$

   $\mathrm{Dim}(W) = 2$ #vertices - 1

   W is a walk on the directed edges (not vertices)

$|p_x\rangle = \sum \sqrt{p_{y\leftarrow x}}\ |y\rangle$ $\quad T\,|x\rangle = |x, p_x\rangle$ $\quad T\,T^\dagger = \sum |x, p_x\rangle\langle x, p_x|$

$S = \mathrm{Swap} = \sum |x, y\rangle\langle y, x|$ $\quad T = \sum |x, p_x\rangle\langle x|$ $\quad T^\dagger T = \mathrm{Id}$

Element distinctness

12	6	7	2	14	10	9	7	20	5	1	16	17

$[N] = \{1, 2, \ldots, N\}$

Algorithm:

- Pick k indices $K \subseteq [N]$
- Query oracle on K
- Repeat
  - Check for collision in K
  - Replace **one** index in $K \subseteq [N]$
  - Query oracle on the new index

$$|\text{final}\rangle = (\mathbf{WG})^{\mathbf{T}} |\mathbf{init}\rangle$$

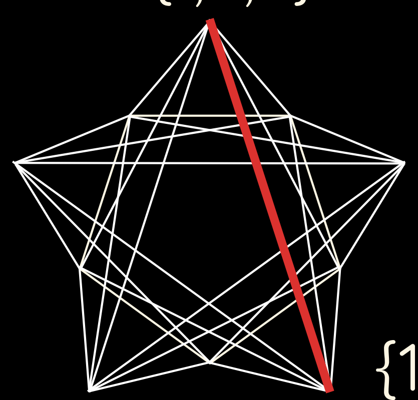
Setup $\mathbf{S} = k$

Check $\mathbf{C} = 0$

Update $\mathbf{U} = 1$

$$\mathrm{HT} = \frac{N^2}{k}$$

$$\begin{aligned} \text{Cost} &= \mathbf{S} + \sqrt{\mathrm{HT}}\,(\mathbf{C} + \mathbf{U}) \\ &= k + \sqrt{N^2/k}\,(0 + 1) \\ &= k + \sqrt{N^2/k} \\ &= N^{2/3} \qquad \text{for } k = N^{2/3} \end{aligned}$$



Non-Disjointness

$a \in \{0,1\}^N$

0	0	1	1	0	1	1	0	**1**	1	0	0

$$\sum |i\rangle|0\rangle$$

$$\sum |i\rangle|a_i\rangle$$

$b \in \{0,1\}^N$

0	1	0	0	1	0	0	0	**1**	0	1	1

$$\sum |i\rangle|a_i\rangle$$

$$\sum (-1)^{a_i \wedge b_i} |i\rangle\, |a_i\rangle$$

$$\sum (-1)^{a_i \wedge b_i} |i\rangle\, |a_i\rangle$$

$$\sum (-1)^{a_i \wedge b_i} |i\rangle\, |0\rangle$$

$\exists$ index i: $a_i = b_i = 1$?

	Bits sent
Trivial	N
Random	$N \log N$
Q Search	$\sqrt{N} \log N$

Non-Disjointness

$a \in \{0,1\}^N$

0	0	1	1	0
1	1	0	**0**	1
0	0	0	1	0
1	0	0	1	1
0	0	0	1	0

cell i

$b \in \{0,1\}^N$

0	1	0	0	1
0	0	0	**1**	0
1	1	0	0	1
0	1	0	0	0
1	0	1	0	0

cell i

my cell contains 1

picks next $\in_R$ {left, above, right, below}

go to the left cell

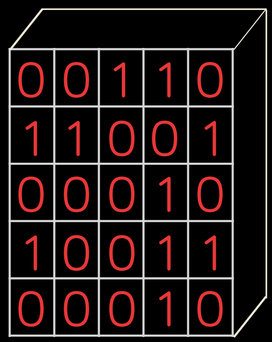
$\exists$ index i: $a_i = b_i = 1$?

	Bits sent
Trivial	N
Random	$N \log N$
Q Search	$\sqrt{N} \log N$
2D-Grid	$N \log N$

Hitting Time on 2D-Grid $= N \log N$

Non-Disjointness

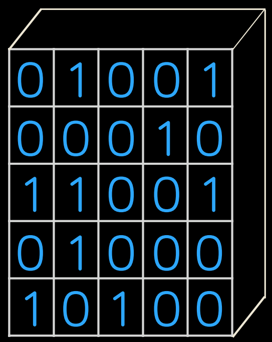
$a \in \{0,1\}^N$



$\sum |i\rangle$

$\sum (-1)^{a_i \wedge b_i} |i\rangle$

$b \in \{0,1\}^N$



$|i\rangle$

$|i\rangle$

$\exists$ index i: $a_i = b_i = 1$?

	Bits sent
Trivial	N
Random	$N \log N$
Q Search	$\sqrt{N} \log N$
2D-Grid	$N \log N$
Q 2D-Grid	$\sqrt{N \log N}$
3D-Grid	N
Q 3D-Grid	$\sqrt{N}$

Non-Disjointness can be solved with $\sqrt{N}$ qubits communicated

Hitting Time on 3D-Grid $= N$

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Thank you