Occam's Vorpal Quantum Razor: Memory Reduction When Simulating Continuous-Time Stochastic Processes With Quantum Devices

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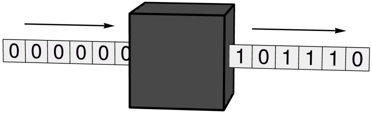
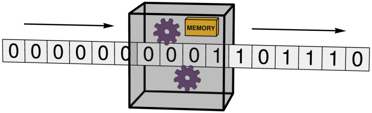
Observation, Modelling and Simulation

Science: Make observations, construct models, and build simulators.





Multiple models can provide identical predictions. How to choose the best?

One philosophy is to follow Occam's Razor:

> *"Plurality should not be posited without necessity"*

Computational Mechanics

What is 'simple'?

Computational Mechanics: Simple processes require less memory.

Observations can be expressed as a time series:

$$\ldots x_{-5}x_{-4}x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3x_4x_5\ldots$$

We divide the process into past $\overleftarrow{x} = x_{-\infty:0}$ and future $\overrightarrow{x} = x_{0:\infty}$.

The causal states of a process are sets of pasts with identical future statistics:

$$P(\overrightarrow{X}|\overleftarrow{X} = \overleftarrow{x}) = P(\overrightarrow{X}|\overleftarrow{X} = \overleftarrow{x}') \Leftrightarrow \overleftarrow{x} \sim_e \overleftarrow{x}'.$$

J. P. Crutchfield and K. Young, Phys. Rev. Lett. **63** 105 (1989)

Computational Mechanics

Example: Random Process.

$$11011001110110100111101111101010111110001010100010011110 1101$$

$$\{*\}.$$

Example: Perturbed Coin

$$0111110001111011111011111110000000100000000111111110000110011$$

$$\{*0\} \qquad \{*1\}.$$

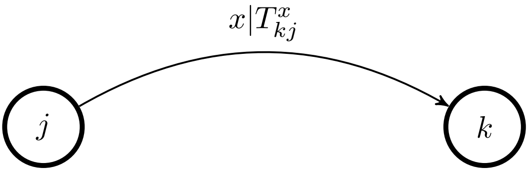
Example: No Triple Zero

$$1101100111011010011110111110101011111001010101001010011111$$

$$\{*00\} \qquad \{*10\} \qquad \{*1\}.$$

Computational Mechanics

The simplest classical model is the ε-machine. It can be represented as an edge-emitting hidden Markov model based on the causal states.



Statistical Complexity: Shannon entropy of the steady-state distribution.

$$C_\mu = -\sum_j \pi(S_j) \log_2(\pi(S_j)).$$

This is the minimum memory required by a classical simulator of the model.

J. P. Crutchfield and K. Young, Phys. Rev. Lett. **63** 105 (1989)
C. R. Shalizi and J. P. Crutchfield, J. Stat. Phys. **104** 817 (2001)

Quantum Computational Mechanics

Even optimal classical models store redundant information. They do not exploit overlap in the future statistics of different causal states.

Quantum mechanics allows information storage in non-orthogonal states, and can mitigate some of this redundancy.

$$|S_j\rangle = \sum_{xk} \sqrt{T^x_{kj}} |x\rangle |k\rangle.$$

We call such constructions q-machines. Their memory requirement is

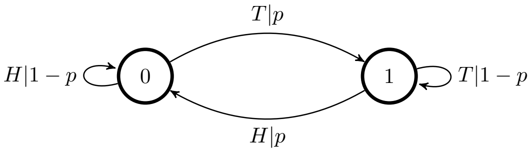
$$C_q = -\mathrm{Tr}(\rho \log_2 \rho) \qquad \rho = \sum_j \pi(S_j) |S_j\rangle\langle S_j|.$$

This is less than that of the corresponding ε-machine: $C_q \leq C_\mu$.

M. Gu, K. Wiesner, E. Rieper, and V. Vedral, Nat. Comm. **3** 762 (2012)

Quantum Computational Mechanics

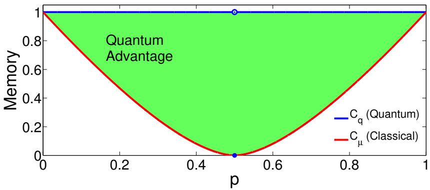
Example: The perturbed coin.



Classically, the memory requirement is almost always 1 bit. Quantum mechanically it is almost always less.

$$|S_0\rangle = \sqrt{p}|H\rangle|0\rangle + \sqrt{1-p}|T\rangle|1\rangle$$

$$|S_1\rangle = \sqrt{1-p}|H\rangle|0\rangle + \sqrt{p}|T\rangle|1\rangle$$



M. Gu, K. Wiesner, E. Rieper, and V. Vedral, Nat. Comm. **3** 762 (2012)
M. S. Palsson *et al.*, Science Advances **3** e1601302 (2017)

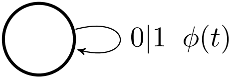
Continuous-Time Stochastic Processes

Continuous-time processes also detail the times between events: $\boldsymbol{x}_n = (x_n, t_n)$.
Process dynamic includes waiting time distributions $\phi^x_{kj}(t)$.

The past now includes the time since the last emission. Likewise for the future.

$$\overleftarrow{\boldsymbol{x}} = \boldsymbol{x}_{-\infty:0}(\emptyset, t_{0^+}) \qquad \overrightarrow{\boldsymbol{x}} = (x_0, t_{0^-})\boldsymbol{x}_{1:\infty}.$$

Renewal Processes: single symbol IID emissions with wait time distribution $\phi(t)$.



Simple, yet broad applicability, e.g. queues, lifetimes, neural spike trains.

The only relevant part of the past is the time since the last emission:

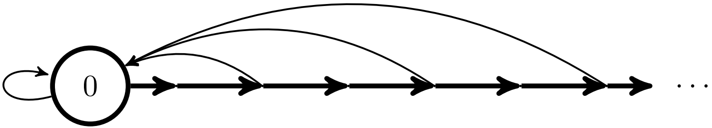
$$t_{0^+} \sim_e t'_{0^+} \Leftrightarrow P(T_{0^-} | T_{0^+} = t_{0^+}) = P(T_{0^-} | T_{0^+} = t'_{0^+}).$$

S. E. Marzen and J. P. Crutchfield, Entropy **17** 4891 (2015)
S. E. Marzen and J. P. Crutchfield, J. Stat. Phys. pp. 1-19 (2017).

Continuous-Time Stochastic Processes

Classically, the causal states of renewal processes are well-characterised.

For non-Poissonian processes the ε-machine tracks the time since last emission.



Continuous-time requires tracking to arbitrary precision. This leads to a divergent memory requirement.

With finite memory, forced to make a trade-off between precision and storage.

S. E. Marzen and J. P. Crutchfield, Entropy **17** 4891 (2015)
S. E. Marzen and J. P. Crutchfield, J. Stat. Phys. pp. 1-19 (2017).

Quantum Simulators of Continuous-Time Processes

What happens when applying quantum treatment to continuous-time processes?

Analogous to discrete-time, define states using conditional probabilities $(\psi(t) = \sqrt{\phi(t)})$:

$$|S_t\rangle = \int_0^\infty dt' \sqrt{P(T_{0^-} = t' | T_{0^+} = t)} |t'\rangle$$
$$= \frac{1}{\sqrt{\Phi(t)}} \int_0^\infty dt' \psi(t + t') |t'\rangle.$$

Statistics encoded into wavefunction, measurement in $\{|t\rangle\}$ gives correct probabilities.

Passage of time mimicked by measurement sweeps along the continuous-variable. Non-detection projects to state with the correct conditional statistics.

T. J. Elliott and M. Gu, arXiv:1704.04231

Quantum Simulators of Continuous-Time Processes

By construction, the quantum causal states will self-merge if two states have identical future statistics.

This allows us to neglect the causal architecture needed in the classical case, and naively construct the corresponding states for all possible wait times.

Different quantum causal states typically have non-zero overlap:

$$\langle S_a | S_b \rangle = \frac{1}{\sqrt{\Phi(a)\Phi(b)}} \int_0^\infty dt \psi(t+a)\psi(t+b).$$

Thus, the q-machine requires less memory than the corresponding ε-machine.

T. J. Elliott and M. Gu, arXiv:1704.04231

Quantum Simulators of Continuous-Time Processes

The q-machine memory requirement can be found from the steady state.

Eigenvalue equation for steady-state density matrix:

$$\mu \int_0^\infty db \int_0^\infty dt \psi(t+a)\psi(t+b) f_n(b) = \lambda_n f_n(a).$$

Solving this, we have

$$C_q = -\sum_n \lambda_n \log_2 \lambda_n.$$

This is insensitive to time rescaling.

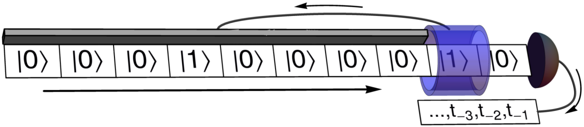
T. J. Elliott and M. Gu, arXiv:1704.04231

Quantum Simulators of Continuous-Time Processes

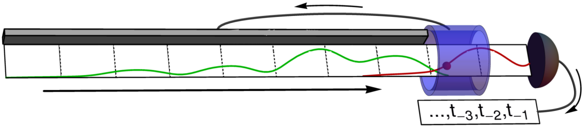
Digital simulation is easy due to IID renewal processes. Prepare $|S_{t_{0^+}}\rangle|S_0\rangle^{\otimes L-1}$.

Analogue simulation is much more interesting!

Discretised version: Prepare qubits in $|\sigma_t\rangle$. Sweep along chain, and conditionally transform $|1\rangle|0\rangle^{\otimes\infty} \to |1\rangle|\sigma_0\rangle$. Measurement of $|1\rangle$ signifies emission event.



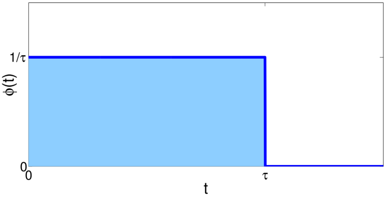
Continuous: Prepare particle in $|S_t\rangle$. Gate conditioned on particle presence.

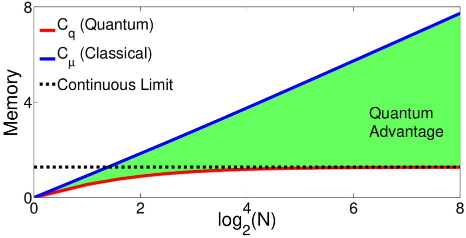


T. J. Elliott and M. Gu, arXiv:1704.04231

Quantum Simulators of Continuous-Time Processes

Example: Uniform Emission Probability.





$$\lambda_n = \frac{8}{(\pi(2n-1))^2}$$

$$C_q \approx 1.2809$$

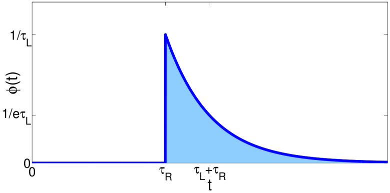
q-machine memory is finite, but C_μ diverges - unbounded advantage![1]

T. J. Elliott and M. Gu, arXiv:1704.04231

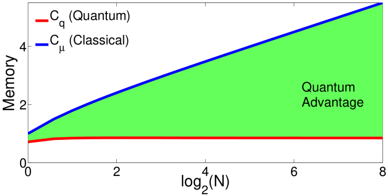
[1] Note also: A. J. P. Garner *et al.*, arXiv:1609.04408 (*to appear in New. J. Phys.*)

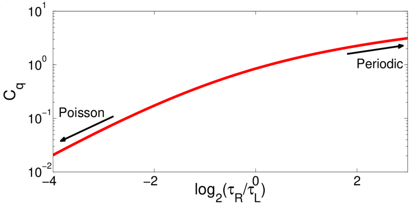
Quantum Simulators of Continuous-Time Processes

Example: Delayed Poisson Process.



Can sweep between pure Poisson and pure periodic processes.





Again an unbounded memory advantage for the *q*-machine - we suspect this may be a typical property.

T. J. Elliott and M. Gu, arXiv:1704.04231

Summary and Outlook

Computational Mechanics provides the tools to determine the most memory-efficient models. Quantum mechanics allows classical bounds on memory to be beaten.

Classically an unbounded memory is needed to track continuous-time processes to arbitrary precision.

Quantum mechanics reduces this memory requirement, and arbitrary precision can be achieved with finite memory.

Future Questions:

- Proof of conjecture about typicality of bounded memory?
- Extension to more general continuous-time stochastic processes.
- Experimental implementation.
- Advantages in other information-theoretic quantities of q-machines.
- Further optimisation of q-machines.

Thanks for listening!

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T. J. Elliott and M. Gu, arXiv:1704.04231