Semidefinite programming converse bounds for quantum communication

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Joint work with Xin Wang, Runyao Duan

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Quantum communication

\[ \approx \text{id} \]

Channel distance \[ \| D \circ N \circ E - \text{id} \| \]

Channel fidelity \[ F(\Phi_k, D \circ N \circ E(\Phi_k)) \]

where \( \Phi_k \) is \( k \)-dimensional maximally entangled state.

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Quantum communication

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How well the simulation is? [Kretschmann, Werner, 2004]

- Channel distance $\| \mathcal{D} \circ \mathcal{N} \circ \mathcal{E} - id_k \|_\diamondsuit$.
- Channel fidelity $F(\Phi_k, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\Phi_k))$, where $\Phi_k$ is $k$-dimensional maximally entangled state.

...
Quantum capacity

- $r$: qubits transmitted per channel use.
- $n$: number of channel copies.
- $\varepsilon$: error tolerance.

A triplet $(r, n, \varepsilon)$ is achievable if $\exists \Phi_k, \mathcal{E}_n$ and $\mathcal{D}_n$ such that

$$\frac{1}{n} \log k \geq r, \quad F(\Phi_k, \tilde{\Phi}_k) \geq 1 - \varepsilon.$$ 

Optimal achievable rate given $n, \varepsilon$

$$r^*(n, \varepsilon) := \max \{r : (r, n, \varepsilon) \text{ achievable}\}.$$ 

Quantum capacity

$$Q(N) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} r^*(n, \varepsilon).$$

For any quantum channel \( \mathcal{N} \), its quantum capacity is equal to the regularized coherent information of the channel:

\[
Q(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} I_c(\mathcal{N}^\otimes n),
\]

where \( I_c(\mathcal{N}) = \max_{\phi_{AA'}} I(A_B)_{\mathcal{N}_{A' \to B} (\phi_{AA'})} \) and \( \phi_{AA'} \) pure state.

- Not a single-letter formula.
- \( I_c(\mathcal{N}) \) not additive in general.
## Known converse bounds

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<td>$R$</td>
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<tr>
<td>$E_C$</td>
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<td>$Q_E$</td>
<td>✓</td>
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</tr>
<tr>
<td>$Q_\Theta$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

- **$R$:** Rains information [Tomamichel, Wilde, Winter, 2017]
- **$\varepsilon$-DEG:** Epsilon degradable bound [Sutter, Scholz, Winter, Renner, 2014]
- **$E_C$:** Channel’s entanglement cost [Berta, Brandão, Christandl, Wehner, 2013]
- **$Q_E$:** Entanglement assisted quantum capacity [Bennett, Devetak, Harrow, Shor, Winter, 2014; Berta, Christandl, Renner, 2011]
- **$Q_{ss}$:** Quantum capacity with symmetric side channels [Smith, Smolin, Winter, 2008]
- **$Q_\Theta$:** Partial transposition bound [Holevo, Werner, 2001]
One-shot quantum capacity
One-shot quantum capacity

Unassisted code (UA): \[ \Pi_{A_i B_i \to A_o B_o} = \epsilon_{A_i} \rightarrow A_o \otimes D_{B_i} \rightarrow B_o. \]

Positive partial transpose preserving (PPT) code:

Non-signalling (NS) code:

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One-shot quantum capacity

Unassisted code (UA):

\[ \Pi_{A_i B_i \rightarrow A_0 B_0} = \mathcal{E}_{A_i \rightarrow A_0} \otimes \mathcal{D}_{B_i \rightarrow B_0}. \]

\[ J_{\Pi} = \Pi_{A_i B_i \rightarrow A_0 B_0} \left( \Phi_{A_i B_i; A'_i B'_i} \right) \]
One-shot quantum capacity

- Unassisted code (UA):
  \[
  \Pi_{A_iB_i \rightarrow A_oB_o} = \mathcal{E}_{A_i \rightarrow A_o} \otimes \mathcal{D}_{B_i \rightarrow B_o}.
  \]

- Positive partial transpose preserving (PPT) code: [Rains, 1999; Rains, 2001]
  \[
  \Pi_{A_iB_i \rightarrow A_oB_o} \text{ PPT operation } J_{\Pi}^{T_{B_iB_o}} \geq 0.
  \]

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One-shot quantum capacity

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Positive partial transpose preserving (PPT) code: [Rains, 1999; Rains, 2001]

\[ \Pi_{A_iB_i \to A_0B_0} \text{ PPT operation } J_{\Pi}^{T_{B_iB_0}} \geq 0. \]

Non-signalling (NS) code: [Leung, Matthews, 2015; Duan, Winter, 2016]

\[ \text{Tr}_{A_0} J_{\Pi} = \frac{1}{d_{A_i}} \otimes \text{Tr}_{A_iA_0} J_{\Pi}, \quad (A \rightarrow B) \]
\[ \text{Tr}_{B_0} J_{\Pi} = \frac{1}{d_{B_i}} \otimes \text{Tr}_{B_iB_0} J_{\Pi}, \quad (B \rightarrow A) \]

\[ J_{\Pi} = \Pi_{A_iB_i \to A_0B_0} \left( \Phi_{A_iB_i; A_i'B_i} \right) \]
One-shot quantum capacity

Unassisted code (UA):

$$\Pi_{A_iB_i \rightarrow A_0B_0} = \mathcal{E}_{A_i \rightarrow A_0} \otimes \mathcal{D}_{B_i \rightarrow B_0}. $$

Positive partial transpose preserving (PPT) code: [Rains, 1999; Rains, 2001]

$$\Pi_{A_iB_i \rightarrow A_0B_0} \text{ PPT operation } J_{\Pi}^{T_{B_0:B_0}} \geq 0.$$ 

Non-signalling (NS) code: [Leung, Matthews, 2015; Duan, Winter, 2016]

$$\text{Tr}_{A_0} J_{\Pi} = \frac{1_{A_i}}{d_{A_i}} \otimes \text{Tr}_{A_iA_0} J_{\Pi}, \quad (A \rightarrow B)$$

$$\text{Tr}_{B_0} J_{\Pi} = \frac{1_{B_i}}{d_{B_i}} \otimes \text{Tr}_{B_iB_0} J_{\Pi}, \quad (B \rightarrow A)$$

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Maximum channel fidelity

$$F_{\Omega}(N, k) := \sup_{\Pi \in \Omega} \text{Tr} \left( \Phi_k \cdot \Pi \circ N(\Phi_k) \right).$$

One-shot quantum capacity

$$Q^{(1)}_{\Omega}(N, \varepsilon) := \log \max \{ k : F_{\Omega}(N, k) \geq 1 - \varepsilon \}.$$

(Asymptotic) quantum capacity

$$Q_{\Omega}(N) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q^{(1)}_{\Omega}(N^\otimes n, \varepsilon).$$
SDP converse bounds for one-shot quantum capacity

[Leung, Matthews, 2015]

\[ F_{\Omega} (N, k) = \max \text{Tr} \, J_N W_{AB} \text{ s.t. } 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \text{Tr} \, \rho_A = 1, \]

**PPT:** \(- k^{-1} \rho_A \otimes 1_B \leq W_{AB}^{TB} \leq k^{-1} \rho_A \otimes 1_B, \) **NS:** \(\text{Tr}_A W_{AB} = k^{-2} 1_B.\)

**Optimization characterization**

\[ Q_{PPT}^{(1)} (N, \epsilon) = - \log \min m \]

s.t. \( \text{Tr} \, J_N W_{AB} \geq 1 - \epsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)

\( \text{Tr} \, \rho_A = 1, -m \rho_A \otimes 1_B \leq W_{AB}^{TB} \leq m \rho_A \otimes 1_B, \)

\[ [\text{Tr}_A W_{AB} = m^2 1_B. \text{ NS condition}] \]
[Leung, Matthews, 2015]

\[ F_\Omega (N, k) = \max \text{Tr} J_N W_{AB} \text{ s.t. } 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \text{Tr} \rho_A = 1, \]

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\[ Q_{PPT}^{(1)} (N, \epsilon) = -\log \min m \]

\[ \text{s.t. } \text{Tr} J_N W_{AB} \geq 1 - \epsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \]

\[ \text{Tr} \rho_A = 1, -m \rho_A \otimes 1_B \leq W_{AB}^T \leq m \rho_A \otimes 1_B, \]

\[ \left[ \text{Tr}_A W_{AB} = m^2 1_B \text{ NS condition} \right] \]

Non-linear terms
\[ Q^{(1)}_{PPT}(N, \varepsilon) = -\log \min m \]
\[
\text{s.t. } \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \\
\text{Tr} \rho_A = 1, -m \rho_A \otimes 1_B \leq W_{AB}^T \leq m \rho_A \otimes 1_B. \\
[\text{Tr}_A W_{AB} = m^2 1_B. \text{ NS condition}] 
\]
\[ Q_{PPT}^{(1)}(N, \varepsilon) = -\log \min m \]

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \)

\[ \text{Tr} \rho_A = 1, -m \rho_A \otimes \mathbb{1}_B \leq W_{AB}^{TB} \leq m \rho_A \otimes \mathbb{1}_B. \]

[\( \text{Tr}_A W_{AB} = m^2 \mathbb{1}_B. \) NS condition]

\[ g(N, \varepsilon) := \min \text{Tr} S_A \]

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \)

\[ \text{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{TB} \leq S_A \otimes \mathbb{1}_B. \]
\[ Q_{PPT}^{(1)}(N, \varepsilon) = -\log \min m \]

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)
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\[ [\text{Tr}_A W_{AB} = m^2 1_B. \text{ NS condition}] \]

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s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)
\( \text{Tr} \rho_A = 1, -S_A \otimes 1_B \leq W_{AB} \leq S_A \otimes 1_B. \)

\[ \tilde{g}(N, \varepsilon) := \min \text{Tr} S_A \]

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)
\( \text{Tr} \rho_A = 1, -S_A \otimes 1_B \leq W_{AB} \leq S_A \otimes 1_B, \)
\( \text{Tr}_A W_{AB} = t 1_B. \)
\[ Q_{PPT}^{(1)} (N, \varepsilon) = -\log \min m \]

\[
\begin{align*}
\text{s.t. } & \text{ Tr } J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \\
& \text{ Tr } \rho_A = 1, -m \rho_A \otimes 1_B \leq W_{AB}^{TB} \leq m \rho_A \otimes 1_B. \\
& [\text{ Tr}_A W_{AB} = m^2 1_B. \text{ NS condition}] 
\end{align*}
\]

\[ g (N, \varepsilon) := \min \text{ Tr } S_A \]

\[
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& \text{ Tr } \rho_A = 1, -S_A \otimes 1_B \leq W_{AB}^{TB} \leq S_A \otimes 1_B, \\
& \text{ Tr}_A W_{AB} = t 1_B. 
\end{align*}
\]

\[ \hat{g} (N, \varepsilon) := \min \text{ Tr } S_A \]

\[
\begin{align*}
\text{s.t. } & \text{ Tr } J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \\
& \text{ Tr } \rho_A = 1, -S_A \otimes 1_B \leq W_{AB}^{TB} \leq S_A \otimes 1_B, \\
& \text{ Tr}_A W_{AB} = t 1_B, t \geq \hat{m}^2, \\
& (Q_{PPT \cap NS}^{(1)} (N, \varepsilon) \leq -\log \hat{m}). 
\end{align*}
\]
Main result 1: SDP converse bounds for one-shot quantum capacity

[Semidefinite programming converse bounds for quantum communication](1709.00200) X. Wang, K. Fang, R. Duan

[Semidefinite programming converse bounds for quantum communication](1709.00200) X. Wang, K. Fang, R. Duan

\[ f(N, \varepsilon) = \min \text{Tr} S_A \]
\[ \text{s.t. } \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, S_A, \Theta_{AB} \geq 0, \text{Tr} \rho_A = 1, \]
\[ 0 \leq W_{AB} \leq \rho_A \otimes 1_B, S_A \otimes 1_B \geq W_{AB} + \Theta_{AB}^{TB}. \]
Main result 1: SDP converse bounds for one-shot quantum capacity

[Tomamichel, Berta, Renes, 2016]

\[
f (N, \varepsilon) = \min \ \text{Tr} \ S_A
\]

\[\text{s.t. } \text{Tr} \ J_N W_{AB} \geq 1 - \varepsilon, S_A, \Theta_{AB} \geq 0, \text{Tr} \ \rho_A = 1, \]

\[0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, S_A \otimes \mathbb{1}_B \geq W_{AB} + \Theta_{AB}^T. \quad (5)\]

**Theorem**

For any quantum channel \( N \) and error tolerance \( \varepsilon \), the inequality chain holds

\[
Q^{(1)} (N, \varepsilon) \leq Q_{PPT \cap NS}^{(1)} (N, \varepsilon) \leq - \log \tilde{g} (N, \varepsilon) \leq - \log \tilde{\tilde{g}} (N, \varepsilon) \leq - \log g (N, \varepsilon) \leq - \log f (N, \varepsilon). \quad (6)
\]
Example: Amplitude damping channel

Amplitude damping channel $\mathcal{N}_{AD} = \sum_{i=0}^{1} E_i \cdot E_i^\dagger$ with 

$$
E_0 = |0\rangle \langle 0| + \sqrt{1 - r} |1\rangle \langle 1| \quad E_1 = \sqrt{r} |0\rangle \langle 1|, \quad 0 \leq r \leq 1
$$
Qubit depolarizing channel $\mathcal{N}_D(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$, where $X, Y, Z$ are Pauli matrices.
Asymptotic quantum capacity
$$Q^{(1)}_{PPT}(N, \varepsilon) = -\log \min m$$

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)

\( \text{Tr} \rho_A = 1, -m\rho_A \otimes 1_B \leq W_{AB}^{TB} \leq m\rho_A \otimes 1_B. \)
SDP strong converse bound for quantum capacity

\[ Q_{PPT}^{(1)} (N, \varepsilon) = -\log \min m \]

s.t. \( \text{Tr} J_N W_{AB} \geq 1 - \varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \)

\[ \text{Tr} \rho_A = 1, -m \rho_A \otimes 1_B \leq W_{AB}^{TB} \leq m \rho_A \otimes 1_B. \]

Take \( R_{AB} = W_{AB} / m \) and throw away the condition \( W_{AB} \leq \rho_A \otimes 1_B \), we obtain an additive SDP upper bound \( Q_{PPT}^{(1)} (N, \varepsilon) \leq Q_\Gamma (N) - \log (1 - \varepsilon), \) where

\[ Q_\Gamma (N) = \log \max \text{Tr} J_N R_{AB} \]

s.t. \( R_{AB}, \rho_A \geq 0, \text{Tr} \rho_A = 1, \)

\[ - \rho_A \otimes 1_B \leq R_{AB}^{TB} \leq \rho_A \otimes 1_B. \]

(7)

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\[ Q_{\text{PPT}}^{(1)} (N, \varepsilon) = - \log \min m \]

\[
\text{s.t. } \begin{align*}
\text{Tr } J_N W_{AB} & \geq 1 - \varepsilon, \quad 0 \leq W_{AB} \leq \rho_A \otimes 1_B, \\
\text{Tr } \rho_A & = 1, \quad -m \rho_A \otimes 1_B \leq W_{AB}^T \leq m \rho_A \otimes 1_B.
\end{align*}
\]

Take \( R_{AB} = W_{AB}/m \) and throw away the condition \( W_{AB} \leq \rho_A \otimes 1_B \), we obtain an additive SDP upper bound

\[
Q_{\text{PPT}}^{(1)} (N, \varepsilon) \leq Q_{\Gamma} (N) - \log (1 - \varepsilon),
\]

where

\[ Q_{\Gamma} (N) = \log \max \text{Tr } J_N R_{AB} \]

\[
\text{s.t. } R_{AB}, \rho_A \geq 0, \text{Tr } \rho_A = 1, \\
\quad \rho_A \otimes 1_B \leq R_{AB}^T \leq \rho_A \otimes 1_B.
\]  

- Additivity: \( Q_{\Gamma} (N \otimes M) = Q_{\Gamma} (N) + Q_{\Gamma} (M) \) (by utilizing SDP duality).
- Converse bound for \( Q (N) \) : \( Q (N) \leq Q_{\text{PPT}} (N) \leq Q_{\Gamma} (N) \).
- For noiseless quantum channel \( J_d \), \( Q (J_d) = Q_{\Gamma} (J_d) = \log_2 d \).
- Strong converse: denote the n-shot optimal rate as \( r \), then \( (r, n, \varepsilon) \) satisfies

\[
nr \leq n Q_{\Gamma} (N) - \log (1 - \varepsilon),
\]

which implies \( \varepsilon \geq 1 - 2^n (Q_{\Gamma} (N) - r) \).
Main result 2: SDP strong converse bound for quantum capacity

Theorem (SDP strong converse bound for Q)

For any quantum channel $N$,

$$Q(N) \leq Q_\Gamma(N) = \log \max \text{Tr } J_N R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0$, $\text{Tr } \rho_A = 1$,

$$-\rho_A \otimes 1_B \leq R_{AB}^T \leq \rho_A \otimes 1_B.$$ 

The fidelity of transmission goes to zero if the rate exceeds $Q_\Gamma(N)$.
Main result 2: SDP strong converse bound for quantum capacity

Theorem (SDP strong converse bound for Q)

For any quantum channel $\mathcal{N}$,

$$Q(\mathcal{N}) \leq Q_{\Gamma}(\mathcal{N}) = \log \max \text{Tr} J_N R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0, \text{Tr} \rho_A = 1,$

$$-\rho_A \otimes 1_B \leq R_{AB}^{T_B} \leq \rho_A \otimes 1_B.$$ 

The fidelity of transmission goes to zero if the rate exceeds $Q_{\Gamma}(\mathcal{N})$.

How to understand $Q_{\Gamma}(\mathcal{N})$?

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Main result 2: SDP strong converse bound for quantum capacity

**Theorem (SDP strong converse bound for Q)**

For any quantum channel $\mathcal{N}$,

$$Q(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) = \log \max \text{Tr} J_\mathcal{N} R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0, \text{Tr} \rho_A = 1$,

$$-\rho_A \otimes 1_B \leq R^B_{AB} \leq \rho_A \otimes 1_B.$$

The fidelity of transmission goes to zero if the rate exceeds $Q_\Gamma(\mathcal{N})$.

**How to understand $Q_\Gamma(\mathcal{N})$?**

$$Q_\Gamma(\mathcal{N}) = \max_{\rho_A \in S(A)} E_W(\mathcal{N}_A' \rightarrow B(\phi_{AA'}))$$

$$= \max_{\rho \in S(A)} \min_{\sigma \in \text{PPT}' } D_{\max}(\mathcal{N}_A' \rightarrow B(\phi_{AA'}), \sigma)$$

where $E_W(\rho) := \log \max \{\text{Tr} \rho R_{AB} : -1_{AB} \leq R^B_{AB} \leq 1_{AB}, R_{AB} \geq 0\}$, [Wang, Duan, 2016], $\phi_{AA'}$ is a purification of $\rho_A$ and PPT' = $\{\sigma \geq 0 : \|\sigma^T_B\|_1 \leq 1\}$. 
Main result 2: SDP strong converse bound for quantum capacity

**Theorem (SDP strong converse bound for Q)**

For any quantum channel $N$,

$$Q(N) \leq Q\Gamma(N) = \log \max \text{Tr} \ J_N R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0, \text{Tr} \ \rho_A = 1,$

$$-\rho_A \otimes 1_B \leq R_{AB}^T \leq \rho_A \otimes 1_B.$$  

The fidelity of transmission goes to zero if the rate exceeds $Q\Gamma(N)$.

**How to understand $Q\Gamma(N)$?**

Entanglement measure

$$Q\Gamma(N) = \max_{\rho_A \in S(A)} E_W(N_{A'\rightarrow B}(\phi_{AA'}))$$

$$= \max_{\rho \in S(A)} \min_{\sigma \in \text{PPT}'} D_{\max} (N_{A'\rightarrow B}(\phi_{AA'})) \| \sigma)$$

where $E_W(\rho) := \log \max \{ \text{Tr} \ \rho R_{AB} : -1_{AB} \leq R_{AB}^T \leq 1_{AB}, R_{AB} \geq 0 \}$, [Wang, Duan, 2016], $\phi_{AA'}$ is a purification of $\rho_A$ and $\text{PPT}' = \{ \sigma \geq 0 : \|\sigma^T\|_1 \leq 1 \}$.

**Remark:** For any EB channel $N$, $Q\Gamma(N) = 0$. If $Q_E(N) \neq 0$, $Q\Gamma(N) < Q_E(N)$.

Semidefinite programming converse bounds for quantum communication(1709.00200)

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Comparison with other bounds

Rains information [Tomamichel, Wilde, Winter, 2016]

$$R(N) := \max_{\rho \in S(A)} \min_{\sigma \in \text{PPT}'} D(N_{A'} \rightarrow B(\phi_{AA'}) \| \sigma)$$

Due to the fact that $D(\rho \| \sigma) \leq D_{\max}(\rho \| \sigma)$ [Datta], we have $R(N) \leq Q_{\Gamma}(N)$.

\[\bigcirc\]

Rains information strongly converse but not known to be efficiently computable in general.

\[\bigcirc\]

$Q_{\Gamma}(N)$ strongly converse and efficiently computable in general.
Comparison with other bounds

Rains information [Tomamichel, Wilde, Winter, 2016]

\[ R(N) := \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \text{PPT}} D(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \sigma) \]

\[ Q_{\Gamma}(N) = \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \text{PPT}} D_{\text{max}}(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \sigma) \]
Comparison with other bounds

Rains information [Tomamichel, Wilde, Winter, 2016]

\[ R(N) := \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \text{PPT}'} D(\mathcal{N}_{A' \rightarrow B}(\phi_{AA'}) \| \sigma) \]

\[ Q_{\Gamma}(N) = \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \text{PPT}'} D_{\max}(\mathcal{N}_{A' \rightarrow B}(\phi_{AA'}) \| \sigma) \]

Due to the fact that \( D(\rho \| \sigma) \leq D_{\max}(\rho \| \sigma) \) [Datta, 2009], we have \( R(N) \leq Q_{\Gamma}(N) \).
Comparison with other bounds

Rains information [Tomamichel, Wilde, Winter, 2016]

\[ R(N) := \max_{\rho \in S(A)} \min_{\sigma \in \text{PPT}'} D(N_{A'} \to B(\phi_{AA'}) \| \sigma) \]

\[ Q_{\Gamma}(N) = \max_{\rho \in S(A)} \min_{\sigma \in \text{PPT}'} D_{\text{max}}(N_{A'} \to B(\phi_{AA'}) \| \sigma) \]

Due to the fact that \( D(\rho \| \sigma) \leq D_{\text{max}}(\rho \| \sigma) \) [Datta, 2009], we have \( R(N) \leq Q_{\Gamma}(N) \).

- \( R(N) \) strong converse but not known to be efficiently computable in general.
- \( Q_{\Gamma}(N) \) strong converse and **efficiently computable** in general.

Semidefinite programming converse bounds for quantum communication(1709.00200)

X. Wang, K. Fang, R. Duan
Comparison with other bounds

○ Partial Transposition bound [Holevo, Werner, 2001]

\[ Q(N) \leq Q_\Theta(N) = \log \|N \circ T\|, \]

where \( T \) is the transpose map, \( \|N\| = \|N \otimes id\|_1 \) and can be characterized by SDP from [Watrous, 2012].
Comparison with other bounds

- Partial Transposition bound [Holevo, Werner, 2001]

\[ Q(N) \leq Q_{\Theta}(N) = \log \|N \circ T\|_\Diamond, \]

where \(T\) is the transpose map, \(\|N\|_\Diamond = \|N \otimes id\|_1\) and can be characterized by SDP from [Watrous, 2012].

Improved efficiently computable bound

For any quantum channel \(N\), it holds \(Q_T(N) \leq Q_{\Theta}(N)\).
Comparison with other bounds

- Partial Transposition bound [Holevo, Werner, 2001]

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where \( T \) is the transpose map, \( \|N\|_\Diamond = \|N \otimes id\|_1 \) and can be characterized by SDP from [Watrous, 2012].

**Improved efficiently computable bound**

For any quantum channel \( N \), it holds \( Q_\Gamma(N) \leq Q_\Theta(N) \).

**Example:** \( N_r = \sum_i E_i \cdot E_i^\dagger \) where \( E_0 = |0\rangle\langle 0| + \sqrt{r}|1\rangle\langle 1| \), \( E_1 = \sqrt{1-r}|0\rangle\langle 1| + |1\rangle\langle 2| \).

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Converse bounds comparison

For any quantum channel \( N \), it holds \( Q(N) \leq R(N) \leq Q_\Gamma(N) \leq Q_\Theta(N) \).
## Known converse bounds

<table>
<thead>
<tr>
<th></th>
<th>Strong converse</th>
<th>Efficiently computable</th>
<th>For general channels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_\Gamma$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$R$</td>
<td>✓</td>
<td>✓ (max-min)</td>
<td>✓</td>
</tr>
<tr>
<td>$\varepsilon$-DEG</td>
<td>?</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>$E_C$</td>
<td>✓</td>
<td>? (regularization)</td>
<td>✓</td>
</tr>
<tr>
<td>$Q_E$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$Q_{ss}$</td>
<td>?</td>
<td>? (unbounded dimension)</td>
<td>✓</td>
</tr>
<tr>
<td>$Q_\Theta$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

- $Q_\Gamma$: SDP strong converse bound in this talk.
- $R$: Rains information [Tomamichel, Wilde, Winter, 2017]
- $\varepsilon$-DEG: Epsilon degradable bound [Sutter, Scholz, Winter, Renner, 2014]
- $E_C$: Channel’s entanglement cost [Berta, Brandão, Christandl, Wehner, 2013]
- $Q_E$: Entanglement assisted quantum capacity [Bennett, Devetak, Harrow, Shor, Winter, 2014; Berta, Christandl, Renner, 2011]
- $Q_{ss}$: Quantum capacity with symmetric side channels [Smith, Smolin, Winter, 2008]
- $Q_\Theta$: Partial transposition bound [Holevo, Werner, 2001]
- $\exists N$, $Q_\Gamma (N) < \varepsilon$-DEG ($N$).

Semidefinite programming converse bounds for quantum communication(1709.00200)  
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### Theorem (SDP converse bounds for finite blocklength Q)

For any quantum channel $\mathcal{N}$ and error tolerance $\epsilon$, the inequality chain holds

$$Q^{(1)}(\mathcal{N}, \epsilon) \leq Q^{(1)}_{\text{PPT} \cap \text{NS}}(\mathcal{N}, \epsilon) \leq -\log \hat{g}(\mathcal{N}, \epsilon) \leq -\log \tilde{g}(\mathcal{N}, \epsilon) \leq -\log g(\mathcal{N}, \epsilon) \leq -\log f(\mathcal{N}, \epsilon).$$

---

### Theorem (SDP strong converse bound for Q)

For any quantum channel $\mathcal{N}$,

$$Q(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) = \log \max \text{Tr} J_\mathcal{N} R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0$, Tr $\rho_A = 1$,

$$-\rho_A \otimes 1_B \leq R_{AB}^{TB} \leq \rho_A \otimes 1_B.$$ 

$$Q(\mathcal{N}) \leq R(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N}).$$

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Semidefinite programming converse bounds for quantum communication(1709.00200)

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Open questions and future works

- How to apply our relaxation technique to Gaussian channels?
- $Q_1$ does not work well for depolarizing channels. Can we obtain a better result from the linear programs $\hat{g}$, $\tilde{g}$ or $g$?
THE END

THANK YOU!
References


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