Self-testing of binary observables based on commutation

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Outline

- What is nonlocality?
- What is self-testing?
- The CHSH inequality
- The biased CHSH inequality
- Multiple anticommuting observables
- Summary and open problems

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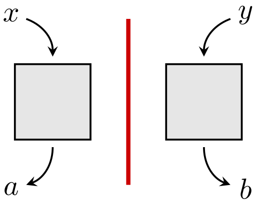
What is nonlocality?

Bell scenario



$\Pr[a, b|x, y]$

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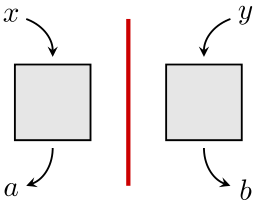

$$\Pr[a, b|x, y]$$

Def.: $\Pr[a, b|x, y]$ is **local** if

$$\Pr[a, b|x, y] = \sum_{\lambda} p(\lambda)\, p(a|x, \lambda)\, p(b|y, \lambda).$$

Otherwise $\implies$ **nonlocal** or it **violates (some) Bell inequality**

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Assume quantum mechanics. . . what can I deduce about my system?

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$$\Pr[a,b|x,y] = \mathrm{tr}\left[(P_a^x \otimes Q_b^y)\rho_{AB}\right] = \sum_{\lambda} p_\lambda \cdot \underbrace{\mathrm{tr}(P_a^x \sigma_\lambda)}_{p(a|x,\lambda)} \cdot \underbrace{\mathrm{tr}(Q_b^y \tau_\lambda)}_{p(b|y,\lambda)}$$

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Given $\Pr[a, b|x, y] = \mathrm{tr}\left[(P_a^x \otimes Q_b^y)\rho_{AB}\right]$

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often only promised some Bell violation

$$\sum_{abxy} c_{ab}^{xy} \Pr[a, b|x, y] = \beta$$

What is self-testing?

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What is self-testing?

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ρ_{AB}

state
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P_a^x, Q_b^y

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Which measurements can be certified?

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measurement certification

Which measurements can be certified?

IN A TRULY ROBUST FASHION...

What is self-testing?

Why care about self-testing of measurements?

- significantly less studied (particularly in the robust regime)
- relevant for (two-party) device-independent cryptography
- pinning down the optimal measurements immediately gives the optimal state

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The CHSH inequality

Measurements with **two outcomes**

$$\begin{aligned} F_j &= F_j^\dagger, \\ F_j &\geq 0, \\ F_0 + F_1 &= \mathbb{1} \end{aligned}$$

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Conveniently written as **observables**

$$A = F_0 - F_1$$

One-to-one mapping, i.e. any

$$A = A^\dagger \quad \text{and} \quad -\mathbb{1} \leq A \leq \mathbb{1}$$

corresponds to a valid measurement
[for projective measurements $A^2 = \mathbb{1}$]

The CHSH inequality

The CHSH value

$$\beta := \operatorname{tr}\left(W \rho_{AB}\right) \quad \text{for} \quad W := A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)$$

Classically $\beta \leq 2$, but quantumly can reach up to $2\sqrt{2}$

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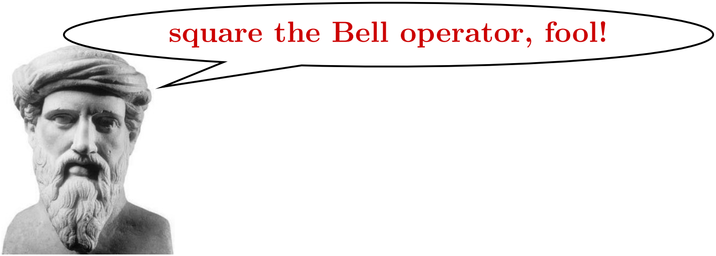

The CHSH inequality

If $A_j^2 = B_k^2 = \mathbb{1}$, then

$$W^2 = 4 \cdot \mathbb{1} \otimes \mathbb{1} - [A_0, A_1] \otimes [B_0, B_1].$$

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In general ($A_j^2, B_k^2 \leq \mathbb{1}$)

$$W^2 \leq 4 \cdot \mathbb{1} \otimes \mathbb{1} - [A_0, A_1] \otimes [B_0, B_1].$$

Simple upper bounds

$$\begin{aligned} W^2 &\leq 4 \cdot \mathbb{1} \otimes \mathbb{1} + |[A_0, A_1] \otimes [B_0, B_1]| \\ &= 4 \cdot \mathbb{1} \otimes \mathbb{1} + |[A_0, A_1]| \otimes |[B_0, B_1]| \\ &\leq 4 \cdot \mathbb{1} \otimes \mathbb{1} + 2|[A_0, A_1]| \otimes \mathbb{1}. \end{aligned}$$

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The Cauchy-Schwarz inequality

$$\big[\,\mathrm{tr}(W\rho_{AB})\big]^2 \leq \mathrm{tr}(W^2\rho_{AB}) \cdot \mathrm{tr}\,\rho_{AB} = \mathrm{tr}(W^2\rho_{AB})$$

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$$\left[\, \mathrm{tr}(W\rho_{AB})\right]^2 \leq \mathrm{tr}(W^2\rho_{AB}) \cdot \mathrm{tr}\,\rho_{AB} = \mathrm{tr}(W^2\rho_{AB})$$

leads to

$$\beta \leq 2\sqrt{1+t},$$

where $t := \frac{1}{2}\,\mathrm{tr}\left(|[A_0, A_1]|\rho_A\right)$.

Bell violation certifies incompatibility of observables!

The CHSH inequality

The quantity

$$t := \frac{1}{2}\,\mathrm{tr}\left(|[A_0, A_1]|\rho_A\right)$$

- invariant under local unitaries and adding auxiliary systems
- easy to compute
- clear operational interpretation as "weighted average"
- $t = 1$ (max. value) implies

$$UA_0U^\dagger = \sigma_x \otimes \mathbb{1},$$
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[assuming ρ_A is full-rank]

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$\Longrightarrow$ $t =$ "**distance from the optimal arrangement**"

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BONUS: $\beta = 2\sqrt{2}$ implies $t = 1$ and so

$$UA_0U^\dagger = \sigma_x \otimes \mathbb{1},$$
$$UA_1U^\dagger = \sigma_y \otimes \mathbb{1}$$

By symmetry the same applies to Bob, so W (up to local unitaries) is just a **two-qubit operator tensored with identity** $\implies$ finding the optimal state is easy

The CHSH inequality

Complete rigidity statement: if $\beta = 2\sqrt{2}$ then there exists $U = U_A \otimes U_B$ and $\tau_{A'B'}$

$$\rho_{AB} = U(\Phi_{AB} \otimes \tau_{A'B'})U^\dagger,$$

where $\Phi_{AB} =$ EPR pair and

$$U_A A_0 U_A^\dagger = \sigma_x \otimes \mathbb{1},$$
$$U_A A_1 U_A^\dagger = \sigma_y \otimes \mathbb{1},$$
$$U_B B_0 U_B^\dagger = \sigma_x \otimes \mathbb{1},$$
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[generalises straightforwardly to multipartite inequalities: Mermin/MABK inequalities]

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For $\alpha \geq 1$ the biased CHSH value

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for

$$W_\alpha := \alpha(A_0 + A_1) \otimes B_0 + (A_0 - A_1) \otimes B_1.$$

Classically $\beta \leq 2\alpha$, but quantumly we can reach up to $2\sqrt{\alpha^2 + 1}$.

- optimal state: maximally entangled of 2 qubits
- optimal observables of Bob: maximally incompatible
- optimal observables of Alice: **non-maximally incompatible!**

The biased CHSH inequality

Analogous argument leads to

$$\beta_\alpha \leq 2\sqrt{\alpha^2 + t_\alpha}$$

for $t_\alpha := \mathrm{tr}(T_\alpha \rho_A)$, where

$$T_\alpha := \frac{\alpha^2 - 1}{4}\big(\{A_0, A_1\} - 2 \cdot \mathbb{1}\big) + \frac{\alpha}{2}|[A_0, A_1]|.$$

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- $t_\alpha = 1$ (max. value) implies

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Any pair of qubit observables can be robustly certified!

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Multiple anticommuting observables

Problem with 3 anticommuting observables: cannot distinguish

$$(\sigma_x, \sigma_y, \sigma_z) \quad \text{vs.} \quad (\sigma_x, -\sigma_y, \sigma_z)$$

[not unitarily equivalent; related by transposition]

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Standard self-testing statement: exists projective observable Υ ($\Upsilon^2 = \mathbb{1}$):

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[direct sum of the two arrangements]

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Not symmetric 

Multiple anticommuting observables

A simple extension of CHSH gives

$$\mathrm{tr}\left(|[A_0, A_1]|\rho_A\right) = \mathrm{tr}\left(|[A_0, A_2]|\rho_A\right) = \mathrm{tr}\left(|[A_1, A_2]|\rho_A\right) = 2$$

[generalises straightforwardly to arbitrary number]

Simple and **symmetric** 

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Good news: **the two are equivalent!**

It is "natural" to formulate self-testing statements in terms of commutation

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Summary

- Commutation-based formulation is convenient: tight self-testing relations from elementary algebra
- For every angle on a qubit there exists a simple (easy to evaluate) **commutation-based function which measures distance to this arrangement**
- Every such arrangement can be **certified in a robust manner**
- Knowing the commutation structure immediately gives a **full rigidity statement**

Open problems

- What about arrangements of observables that "do not fit" into a qubit? E.g. the maximal violation of I_{3322} requires large dimension (in fact, conjectured to be ∞).
  What is the commutation structure of the optimal observables?
- What about observables with more outcomes?
  E.g. Heisenberg-Weyl observables satisfy "twisted commutation relation"
  $$Z_d X_d = \omega X_d Z_d \qquad \left(\omega = e^{2\pi i/d}\right).$$
  Can we find an inequality which certifies precisely this relation?


Yes, Pooh, quantum mechanics is very strange and nobody really understands it but let's talk about it some other day...
So you can really certify quantum systems without trusting the devices at all?
THE END