Irreversibility of Asymptotic Entanglement Manipulation Under PPT Operations

Xin Wang Runyao Duan

UTS: Centre for Quantum Software and Information

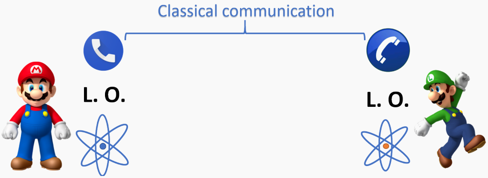
Contain results in arXiv:1606.09421 & 1601.07940

AQIS'17, CQT, Singapore

CENTRE FOR QUANTUM SOFTWARE AND INFORMATION

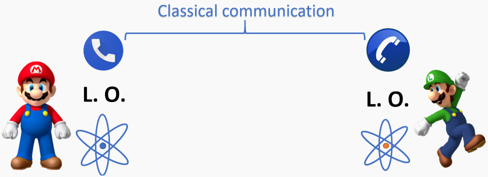
Background

- Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$, it cannot be created by local operations and classical communication (LOCC).



Background

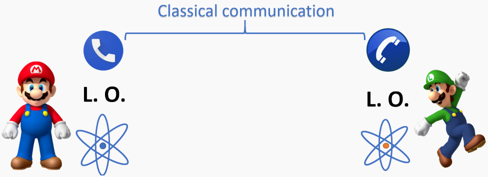
- Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$, it cannot be created by local operations and classical communication (LOCC).



- Entanglement plays a crucial role in quantum physics and is key physical resources in quantum information processing, e.g., super-dense coding, teleportation, etc.

Background

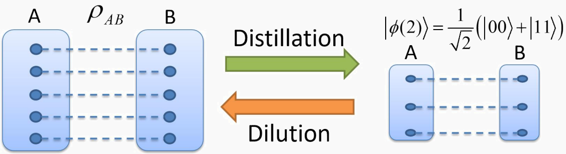
- Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$, it cannot be created by local operations and classical communication (LOCC).



- Entanglement plays a crucial role in quantum physics and is key physical resources in quantum information processing, e.g., super-dense coding, teleportation, etc.
- Entanglement theory studies the detection, quantification, manipulation and applications of entanglement.

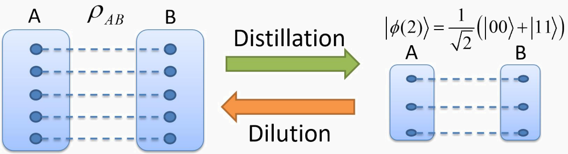
Two fundamental tasks in entanglement manipulations

- **Entanglement distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, 1999, 2001): To extract standard $2\otimes2$ maximally entangled states (Bell states) from a given state ρ by LOCC



Two fundamental tasks in entanglement manipulations

- **Entanglement distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, 1999, 2001): To extract standard 2⊗2 maximally entangled states (Bell states) from a given state ρ by LOCC



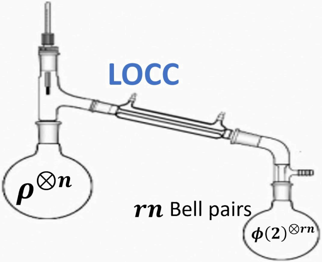
- **Entanglement dilution**: To prepare a given state ρ with the standard Bell states by LOCC

Distillable entanglement

- **Distillable entanglement**: The optimal (maximal) number of Bell states we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \underbrace{\inf_{\Lambda\in\text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1}_{\text{error}} = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.

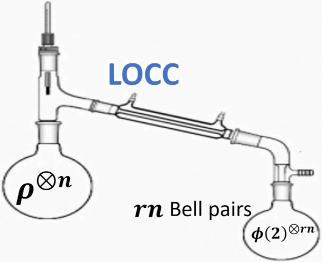


Distillable entanglement

- **Distillable entanglement**: The optimal (maximal) number of Bell states we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \underbrace{\inf_{\Lambda\in\text{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1}_{\text{error}} = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.



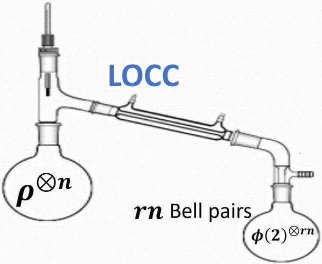
- E_D is hard to evaluate. It is unknown even for isotropic states.

Distillable entanglement

- **Distillable entanglement**: The optimal (maximal) number of Bell states we can extract from ρ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \underbrace{\inf_{\Lambda\in\mathrm{LOCC}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1}_{\text{error}} = 0\}.$$

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.

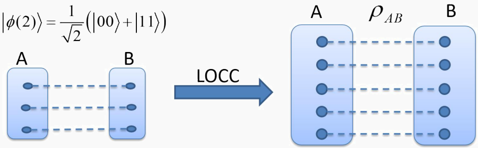


- E_D is hard to evaluate. It is unknown even for isotropic states.
- See next talk (Fang, XW, Tomamichel, Duan, 2017) for non-asymptotic entanglement distillation.

Entanglement cost

- **Entanglement cost**: The optimal (minimal) number of EPR pairs we need to prepare ρ in an asymptotic setting,

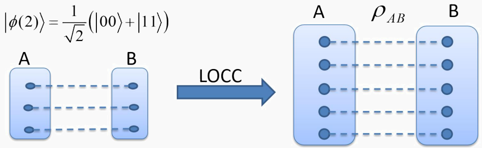
$$E_C(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\text{LOCC}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}.$$



Entanglement cost

- **Entanglement cost**: The optimal (minimal) number of EPR pairs we need to prepare ρ in an asymptotic setting,

$$E_C(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{LOCC}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}.$$



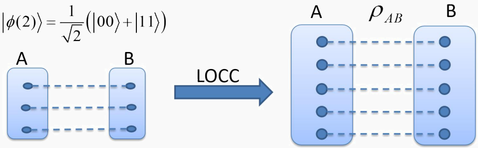
- (Bennett, DiVincenzo, Smolin, Wootters'96) Entanglement of formation is defined by

$$E_F(\rho) := \inf\{\sum_i p_i E(|\psi\rangle\langle\psi|_i) : \rho = \sum_i p_i |\psi\rangle\langle\psi|_i\}.$$

Entanglement cost

- **Entanglement cost**: The optimal (minimal) number of EPR pairs we need to prepare ρ in an asymptotic setting,

$$E_C(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{LOCC}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}.$$



- (Bennett, DiVincenzo, Smolin, Wootters'96) Entanglement of formation is defined by

$$E_F(\rho) := \inf\{\sum_i p_i E(|\psi\rangle\langle\psi|_i) : \rho = \sum_i p_i |\psi\rangle\langle\psi|_i\}.$$

- (Hayden, Horodecki, Terhal'00)

$$E_C(\rho) = \lim_{k\to\infty} \frac{E_F(\rho^{\otimes k})}{k}.$$

Entanglement manipulations and irreversibility

- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.

Entanglement manipulations and irreversibility

- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.
- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\operatorname{Tr}_B |\psi\rangle\langle\psi|).$$

Entanglement manipulations and irreversibility

- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.
- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\operatorname{Tr}_B |\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac'01).

Entanglement manipulations and irreversibility

- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.
- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B\,|\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac'01).
  - In particular, $0 = E_D < E_C$ for any bound entangled states (Yang, Horodecki, Horodecki, Synak-Radtke'05).

Entanglement manipulations and irreversibility

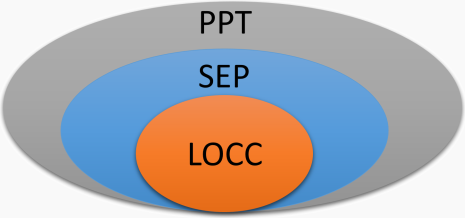
- It is natural to ask whether $E_C \stackrel{?}{=} E_D$.
- Asymptotic entanglement manipulations and irreversibility
  - For **pure states**, asymptotic entanglement manipulation is reversible (Bennett, Bernstein, Popescu, Schumacher'96), i.e.,

$$E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\operatorname{Tr}_B |\psi\rangle\langle\psi|).$$

  - For **mixed states**, this reversibility does not hold any more (Vidal and Cirac'01).
  - In particular, $0 = E_D < E_C$ for any bound entangled states (Yang, Horodecki, Horodecki, Synak-Radtke'05).
- Enlarge the set of operations?

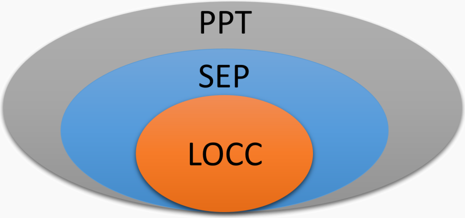
Entanglement manipulations under PPT operations

- One candidate is the set of **PPT operations** (operations completely preserving positivity of partial transpose).



Entanglement manipulations under PPT operations

- One candidate is the set of **PPT operations** (operations completely preserving positivity of partial transpose).

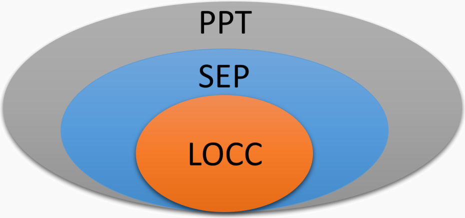


- PPT distillable entanglement (Rains'99)

$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

Entanglement manipulations under PPT operations

- One candidate is the set of **PPT operations** (operations completely preserving positivity of partial transpose).



- PPT distillable entanglement (Rains'99)

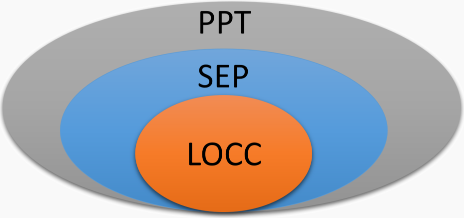
$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

PPT entanglement cost (Audenaert, Plenio, Eisert'03)

$$E_{C,PPT}(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda\in\mathrm{PPT}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}$$

Entanglement manipulations under PPT operations

- One candidate is the set of **PPT operations** (operations completely preserving positivity of partial transpose).



- PPT distillable entanglement (Rains'99)

$$E_{D,PPT}(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_{\Lambda \in \mathrm{PPT}} \|\Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn})\|_1 = 0\}.$$

PPT entanglement cost (Audenaert, Plenio, Eisert'03)

$$E_{C,PPT}(\rho_{AB}) = \inf\{r : \lim_{n\to\infty} \inf_{\Lambda \in \mathrm{PPT}} \|\rho_{AB}^{\otimes n} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}$$

- Clearly, $E_D \le E_{D,PPT} \le E_{C,PPT} \le E_C.$

Reversibility under PPT operations?

- (Audenaert, Plenio, Eisert 2003) The class of antisymmetric states is an example of reversibility under PPT operations.

Reversibility under PPT operations?

- (Audenaert, Plenio, Eisert 2003) The class of antisymmetric states is an example of reversibility under PPT operations.
- (Eggeling, Vollbrecht, Werner, Wolf 2001) For any state ρ with a nonpositive partial transpose, $E_{D,PPT} > 0$.

Reversibility under PPT operations?

- (Audenaert, Plenio, Eisert 2003) The class of antisymmetric states is an example of reversibility under PPT operations.
- (Eggeling, Vollbrecht, Werner, Wolf 2001) For any state ρ with a nonpositive partial transpose, $E_{D,PPT} > 0$.
- **An old open problem** (Audenaert, Plenio, Eisert 2003):

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho)?$$

(The 20^{th} problem listed at the website of Werner's group.)

Reversibility under PPT operations?

- (Audenaert, Plenio, Eisert 2003) The class of antisymmetric states is an example of reversibility under PPT operations.
- (Eggeling, Vollbrecht, Werner, Wolf 2001) For any state ρ with a nonpositive partial transpose, $E_{D,PPT} > 0$.
- **An old open problem** (Audenaert, Plenio, Eisert 2003):

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho)?$$

(The 20^{th} problem listed at the website of Werner's group.)

- A related open problem (Horodecki, Oppenheim, Horodecki 2002):

  Can LOCC+PPT entanglement ensure the reversibility?

Reversibility under PPT operations?

- (Audenaert, Plenio, Eisert 2003) The class of antisymmetric states is an example of reversibility under PPT operations.
- (Eggeling, Vollbrecht, Werner, Wolf 2001) For any state ρ with a nonpositive partial transpose, $E_{D,PPT} > 0$.
- **An old open problem** (Audenaert, Plenio, Eisert 2003):

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho)?$$

(The 20^{th} problem listed at the website of Werner's group.)

- A related open problem (Horodecki, Oppenheim, Horodecki 2002):

  Can LOCC+PPT entanglement ensure the reversibility?

- (Ishizaka, Plenio'05) Multipartite entangled states are not reversible interconvertible under PPT operations, e.g., GHZ and W states.

Main question and outline

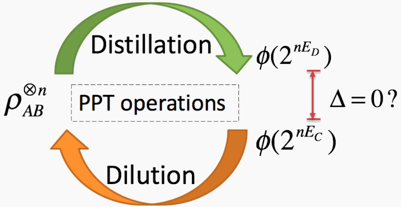
This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?

Main question and outline

This talk is about

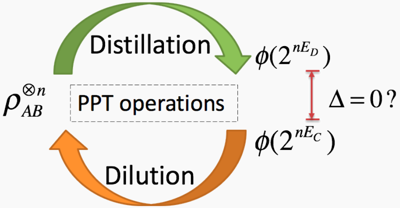
- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



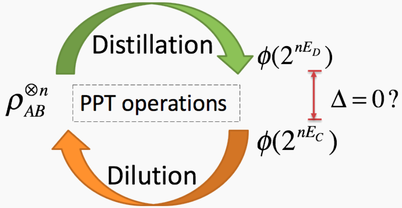
Our approach to resolve the problem:

- Convex optimization (e.g., semidefinite programming).

Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement E_D and entanglement cost E_C?
- Are asymptotic entanglement transformations reversible under PPT operations?



Our approach to resolve the problem:

- Convex optimization (e.g., semidefinite programming).
- Anti-symmetric subspace.

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.
- **Logarithmic negativity** (Vidal, Werner'02):

$$E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1.$$

where T_B means the partial transpose over the system B.

An Upper bound of E_D: Logarithmic negativity

- How to evaluate the distillable entanglement (by any of LOCC, or PPT) is formidable. Only known for very limited cases.
- **Logarithmic negativity** (Vidal, Werner'02):

$$E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1.$$

where T_B means the partial transpose over the system B.

- (Rains, 2001; Vidal and Werner 2002):

$$E_D(\rho_{AB}) \leq E_{D,PPT}(\rho_{AB}) \leq E_N(\rho_{AB}).$$

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log\min\left\{\|X^{T_B}\|_1 : X \geq \rho\right\}.$$

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log \min \left\{ \|X^{T_B}\|_1 : X \geq \rho \right\}.$$

- Properties of E_W:
  i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log \min \left\{ \|X^{T_B}\|_1 : X \geq \rho \right\}.$$

- Properties of E_W:
  i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \leq E_W(\rho).$$

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log\min\left\{\|X^{T_B}\|_1 : X \geq \rho\right\}.$$

- Properties of E_W:
  - **i)** **Additivity**: $E_W(\rho\otimes\sigma) = E_W(\rho) + E_W(\sigma)$.
  - **ii)** **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \leq E_W(\rho).$$

  - **iii)** It is faithful.
  - **iv)** **Non-increasing in average under PPT (LOCC) operations**

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log \min \left\{ \|X^{T_B}\|_1 : X \geq \rho \right\}.$$

- Properties of E_W:

  i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.

  ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \leq E_W(\rho).$$

  iii) It is faithful.

  iv) **Non-increasing in average under PPT (LOCC) operations**

  v) **Improved over logarithmic negativity**: $E_W(\rho) \leq E_N(\rho)$ and the inequality is strict in general.

An improved SDP upper bound of E_D

- Based on the fidelity of PPT distillation (Rains'01), we show an improved SDP upper bound for $E_{D,PPT}$ (XW, Duan'16):

$$E_W(\rho) = \log \min \left\{ \|X^{T_B}\|_1 : X \geq \rho \right\}.$$

- Properties of E_W:
  - i) **Additivity**: $E_W(\rho \otimes \sigma) = E_W(\rho) + E_W(\sigma)$.
  - ii) **Upper bound on PPT distillable entanglement**:

$$E_{D,PPT}(\rho) \leq E_W(\rho).$$

  - iii) It is faithful.
  - iv) **Non-increasing in average under PPT (LOCC) operations**
  - v) **Improved over logarithmic negativity**: $E_W(\rho) \leq E_N(\rho)$ and the inequality is strict in general.
- E_N has all above properties except v)!

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho\log\rho - \rho\log\sigma)$
- PPT relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997)

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \quad \text{s.t.} \quad \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr}\sigma = 1.$$

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho \log \rho - \rho \log \sigma)$
- PPT relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997)

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \ \text{ s.t. } \ \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr} \sigma = 1.$$

- Asymptotic PPT relative entropy of entanglement

$$E^{\infty}_{R,PPT}(\rho) = \lim_{n \to \infty} \frac{1}{n} E_{R,PPT}(\rho^{\otimes n}).$$

Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho \log \rho - \rho \log \sigma)$
- PPT relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997)

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \ \text{ s.t. } \ \sigma, \sigma^{T_B} \geq 0, \operatorname{Tr}\sigma = 1.$$

- Asymptotic PPT relative entropy of entanglement

$$E_{R,PPT}^{\infty}(\rho) = \lim_{n\to\infty} \frac{1}{n} E_{R,PPT}(\rho^{\otimes n}).$$

- (Rains 2001) Rains' bound is the best known upper bound on PPT distillable entanglement, i.e., $E_{D,PPT}(\rho) \leq R(\rho)$, where

$$R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \ \sigma \geq 0, \operatorname{Tr}|\sigma^{T_B}| \leq 1.$$

Nonadditivity of Rains bound & New problem

- (XW, Duan'17) Rains bound is not additive,

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

where the second inequality could be strict, and R^{∞} is the regularized Rains' bound (Hayashi'06).

Nonadditivity of Rains bound & New problem

- (XW, Duan'17) Rains bound is not additive,

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

where the second inequality could be strict, and R^{∞} is the regularized Rains' bound (Hayashi'06).

- **New problem and an old open problem**
  - Note that

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq E_{R,PPT}^{\infty}(\rho) \leq E_{C,PPT}(\rho).$$

  - Do we have $R^{\infty}(\rho) = E_{R,PPT}^{\infty}(\rho)$?

Nonadditivity of Rains bound & New problem

- (XW, Duan'17) Rains bound is not additive,

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq R(\rho),$$

  where the second inequality could be strict, and R^{∞} is the regularized Rains' bound (Hayashi'06).

- **New problem and an old open problem**
  - Note that

$$E_{D,PPT}(\rho) \leq R^{\infty}(\rho) \leq E^{\infty}_{R,PPT}(\rho) \leq E_{C,PPT}(\rho).$$

  - Do we have $R^{\infty}(\rho) = E^{\infty}_{R,PPT}(\rho)$?
  - Dream: if there is a gap, then we will have $E_{D,PPT}(\rho) < E_{C,PPT}(\rho)$!

Nonadditivity of Rains bound & New problem

- (XW, Duan'17) Rains bound is not additive,

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),$$

where the second inequality could be strict, and R^∞ is the regularized Rains' bound (Hayashi'06).

- **New problem and an old open problem**
  - Note that

$$E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq E^\infty_{R,PPT}(\rho) \leq E_{C,PPT}(\rho).$$

  - Do we have $R^\infty(\rho) = E^\infty_{R,PPT}(\rho)$?
  - Dream: if there is a gap, then we will have $E_{D,PPT}(\rho) < E_{C,PPT}(\rho)$!
  - Reality: How to evaluate R^∞ and $E^\infty_{R,PPT}$?

Irreversibility under PPT operations

Theorem (Key result)

There exists entangled state ρ such that $R^\infty(\rho) < E_{R,PPT}^\infty(\rho)$. Thus,

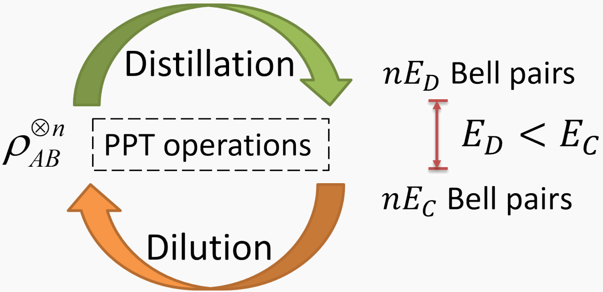
$$\exists\rho,\ \text{s.t.}\ E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$

Irreversibility under PPT operations

Theorem (Key result)

There exists entangled state ρ such that $R^{\infty}(\rho) < E^{\infty}_{R,PPT}(\rho)$. Thus,

$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$



A lower bound of $E_{R,PPT}^{\infty}$

Our key contribution is an efficiently computable lower bound on the regularized relative entropy of entanglement w.r.t. PPT states.

A lower bound for $E_{R,PPT}^{\infty}$

Let P be the projection over the support of state ρ. Then

$$E_{R,PPT}^{\infty}(\rho) \geq E_{\eta}(\rho) = -\log_2 \eta(\rho),$$

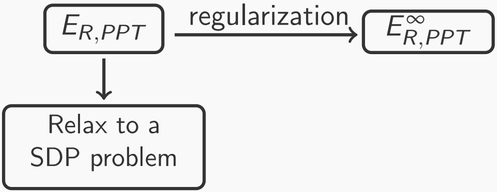
where

$$\eta(\rho) = \min t, s.t. - t\mathbb{1} \leq Y^{T_B} \leq t\mathbb{1}, -Y \leq P^{T_B} \leq Y.$$

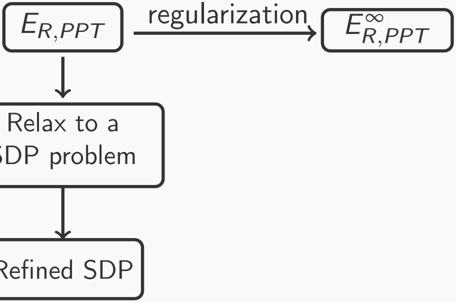
Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

$$\boxed{E_{R,PPT}} \xrightarrow{\text{regularization}} \boxed{E_{R,PPT}^{\infty}}$$

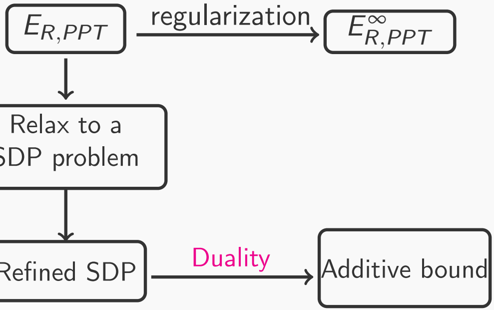
Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof



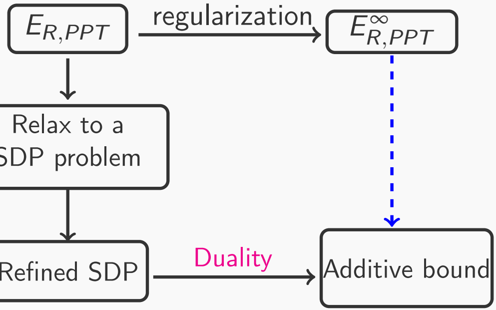
Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof



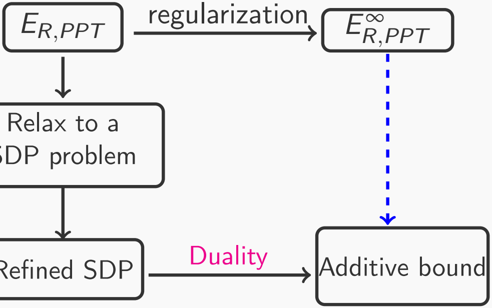
Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof



Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof



Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof

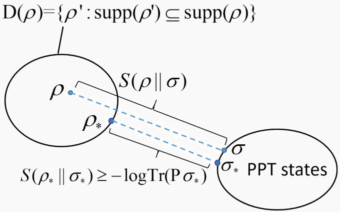


- Relax the problem to an SDP:

$$\min_{\sigma \in PPT} S(\rho\|\sigma) \geq \min_{\rho_* \in D(\rho), \sigma_* \in PPT} S(\rho\|\sigma)$$

$$\geq \min_{\sigma_* \in PPT} -\log \operatorname{Tr} P\sigma_*.$$

Also see min-relative entropy (Datta 2009):
$S(\rho\|\sigma) \geq D_{\min}(\rho\|\sigma) = -\log \operatorname{Tr} P\sigma$



Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof (cont.)

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq -\log\{\min t : Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y\} \quad (\text{not additive } ☹)$$

Lower bound of $E^{\infty}_{R,PPT}$: Sketch of the proof (cont.)

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq -\log\{\min t : Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y\} \quad (\text{not additive} \; ☹)$$

$$\geq \underbrace{-\log\{\min t : \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y\}}_{E_\eta(\rho)}.$$

Lower bound of $E^{\infty}_{R,PPT}$: Sketch of the proof (cont.)

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq -\log\{\min t : Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y\} \quad (\text{not additive} ☹)$$
$$\geq \underbrace{-\log\{\min t : \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y\}}_{E_\eta(\rho)}.$$

- **Prime** / **Dual**

Prime

$$\max \operatorname{Tr} P(V-F)^{T_B}$$
$$\text{s.t. } V + F \leq (W - X)^{T_B}$$
$$\operatorname{Tr}(W+X) \leq 1, V, F, W, X \geq 0.$$

Dual

$$\min\ t$$
$$\text{s.t. } -t\mathbb{1} \leq Y^{T_B} \leq t\mathbb{1},$$
$$-Y \leq P^{T_B} \leq Y.$$

Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof (cont.)

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq -\log\{\min t : Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y\} \quad (\text{not additive} ☹)$$
$$\geq \underbrace{-\log\{\min t : \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y\}}_{E_\eta(\rho)}.$$

- **Prime** / **Dual**

Prime:

$$\max \operatorname{Tr} P(V-F)^{T_B}$$
$$\text{s.t. } V+F \leq (W-X)^{T_B}$$
$$\operatorname{Tr}(W+X) \leq 1, V, F, W, X \geq 0.$$

Dual:

$$\min\ t$$
$$\text{s.t. } -t\mathbb{1} \leq Y^{T_B} \leq t\mathbb{1},$$
$$-Y \leq P^{T_B} \leq Y.$$

- Utilize the strong duality of SDP to obtain

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), \ ☺$$

Lower bound of $E_{R,PPT}^{\infty}$: Sketch of the proof (cont.)

- Utilizing the weak duality of SDP and did a further relaxtion

$$E_{R,PPT}(\rho) \geq -\log\{\min t : Y^{T_B} \leq t\mathbb{1}, P^{T_B} \leq Y\} \quad (\text{not additive } ☹)$$
$$\geq \underbrace{-\log\{\min t : \boxed{-t\mathbb{1} \leq} Y^{T_B} \leq t\mathbb{1}, \boxed{-Y \leq} P^{T_B} \leq Y\}}_{E_\eta(\rho)}.$$

- **Prime** **Dual**

$$\max \operatorname{Tr} P(V-F)^{T_B}$$
$$\text{s.t.}\ V+F \leq (W-X)^{T_B}$$
$$\operatorname{Tr}(W+X) \leq 1, V, F, W, X \geq 0.$$

$$\min\ t$$
$$\text{s.t.} -t\mathbb{1} \leq Y^{T_B} \leq t\mathbb{1},$$
$$-Y \leq P^{T_B} \leq Y.$$

- Utilize the strong duality of SDP to obtain

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), \ ☺$$

thus we have
$$E_{R,PPT}^{\infty}(\rho) \geq \lim_{n\to\infty} \frac{1}{n} E_\eta(\rho^{\otimes n}) = E_\eta(\rho).$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\mathrm{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\mathrm{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\operatorname{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

- Using our established bounds, we have

$$E_{D,PPT}(\rho) = R^{\infty}(\rho) = \log_2(1+\frac{1}{\sqrt{2}}) < 1 = E^{\infty}_{R,PPT}(\rho) = E_{C,PPT}(\rho).$$

Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace

$$\mathrm{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}$$

- **Example 1:** We choose the rank-2 state. Let $\rho = 1/2(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle),$$

- Using our established bounds, we have

$$E_{D,PPT}(\rho) = R^{\infty}(\rho) = \log_2(1+\frac{1}{\sqrt{2}}) < 1 = E_{R,PPT}^{\infty}(\rho) = E_{C,PPT}(\rho).$$

- Any rank-2 state ρ supporting on the $3 \otimes 3$ AS subspace.

Applications

- Efficiently computable lower bound for E_C
- Lower bound for entanglement cost of a quantum channel (Berta, Brandao, Christandl, Wehner'13)

Applications

- Efficiently computable lower bound for E_C
- Lower bound for entanglement cost of a quantum channel (Berta, Brandao, Christandl, Wehner'13)
- Sufficient condition for the irreversibility: If $E_\eta(\rho) > E_W(\rho)$, then

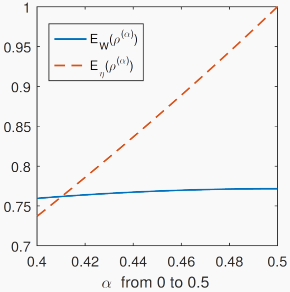
$$E_{D,PPT}(\rho) \le E_W(\rho) < E_\eta(\rho) \le E_{C,PPT}(\rho),$$

Applications

- Efficiently computable lower bound for E_C
- Lower bound for entanglement cost of a quantum channel (Berta, Brandao, Christandl, Wehner'13)
- Sufficient condition for the irreversibility: If $E_\eta(\rho) > E_W(\rho)$, then

$$E_{D,PPT}(\rho) \leq E_W(\rho) < E_\eta(\rho) \leq E_{C,PPT}(\rho),$$

- **Example 2:** For $\rho^{(\alpha)} = (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)/2$ with $|\psi_1\rangle = \sqrt{\alpha}|01\rangle - \sqrt{1-\alpha}|10\rangle$ and $|\psi_2\rangle = \sqrt{\alpha}|02\rangle - \sqrt{1-\alpha}|20\rangle$,



Summary of Results

$$\boxed{E_C \geq E_{C,PPT}} \xrightarrow{\text{SDP approach}} \boxed{E_\eta}$$

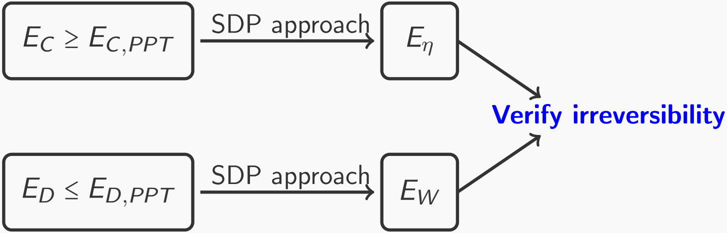
Summary of Results

$$\boxed{E_C \geq E_{C,PPT}} \xrightarrow{\text{SDP approach}} \boxed{E_\eta}$$

$$\boxed{E_D \leq E_{D,PPT}} \xrightarrow{\text{SDP approach}} \boxed{E_W}$$

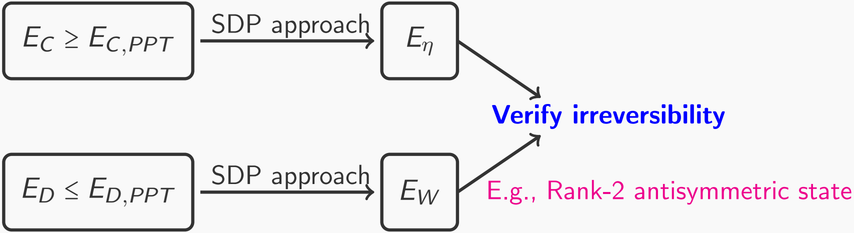
- Better SDP upper bound on E_D & $E_{C,PPT}$
- SDP lower bound for E_C & $E_{C,PPT}$

Summary of Results



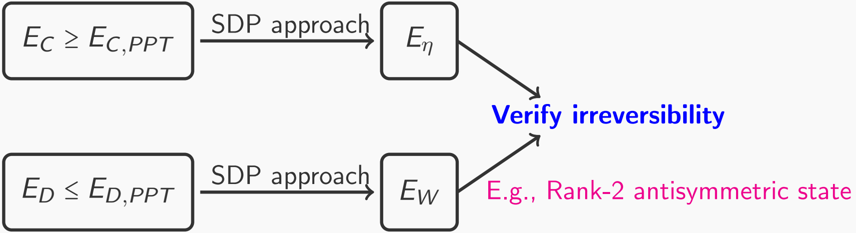
- Better SDP upper bound on E_D & $E_{C,PPT}$
- SDP lower bound for E_C & $E_{C,PPT}$

Summary of Results



- Better SDP upper bound on E_D & $E_{C,PPT}$
- SDP lower bound for E_C & $E_{C,PPT}$

Summary of Results

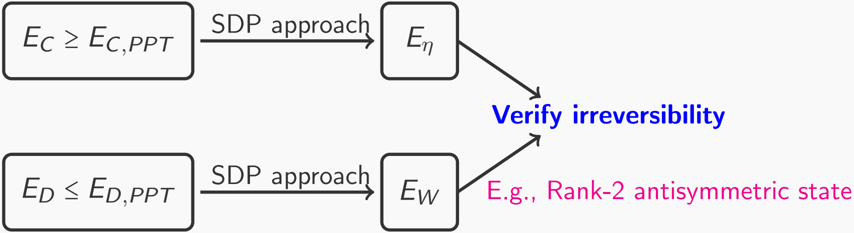


- Better SDP upper bound on E_D & $E_{C,PPT}$
- SDP lower bound for E_C & $E_{C,PPT}$
- Irreversibility under PPT operations:

$$E_{D,PPT} \neq E_{C,PPT},$$

which also holds for LOCC+PPT entanglement.

Summary of Results



- Better SDP upper bound on E_D & $E_{C,PPT}$
- SDP lower bound for E_C & $E_{C,PPT}$
- Irreversibility under PPT operations:

$$E_{D,PPT} \neq E_{C,PPT},$$

  which also holds for LOCC+PPT entanglement.
- There is a gap between Rains' bound and $E^{\infty}_{R,PPT}(\rho)$, which resolves an open problem in (Plenio, Virmani'07).

Outlook

- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling (ANE) operations. Any class of operations between ANE and PPT?

Outlook

- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling (ANE) operations. Any class of operations between ANE and PPT?
- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?

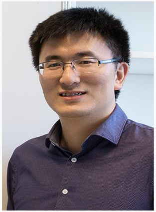
Outlook

- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling (ANE) operations. Any class of operations between ANE and PPT?
- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?
- How to evaluate the distillable entanglement without using PPT operations? NPT bound entanglement?

Outlook

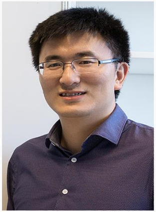
- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling (ANE) operations. Any class of operations between ANE and PPT?
- $E_{D,PPT}(\rho) = R^{\infty}(\rho)$?
- How to evaluate the distillable entanglement without using PPT operations? NPT bound entanglement?
- Any computable upper bound for E_C?
  This will have applications for quantum key repeaters (Baeuml, Christandl, Horodecki, Winter'15).

Details in arXiv: 1606.09421 1601.07940.



Details in arXiv: 1606.09421 1601.07940.

UTS:QSI
CENTRE FOR QUANTUM SOFTWARE AND INFORMATION



Thank you for your attention!