Superadditivity of the Classical Capacity with Limited Entanglement Assistance

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- arXiv: 1704.06955

Quantum Channels and its Capacities

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- An important characteristic is its classical capacity - maximum rate of reliable transmission of classical information

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- Holevo capacity (Holevo '98, Schumacher & Westmorland '97)

$$C^{(1)}(\mathcal{N}) = \max_{\{p_i,\rho_i\}} S\left(\sum p_i \mathcal{N}(\rho_i)\right) - \sum_i p_i S(\mathcal{N}(\rho_i))$$

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- Regularized expression, may be hard to calculate

The Additivity Problem

- The additivity conjecture:

$$C^{(1)}(\mathcal{N}_1 \otimes \mathcal{N}_2) = C^{(1)}(\mathcal{N}_1) + C^{(1)}(\mathcal{N}_2) \quad \forall \mathcal{N}_1, \mathcal{N}_2$$

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- If true, this would mean regularization is not needed, *i.e.*,

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$$C(\mathcal{N}) = C^{(1)}(\mathcal{N}) \quad \forall \mathcal{N}.$$

- However, this turns out to be false :(
- Proven via the subadditivity of minimum output entropy (Hastings '09)

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- Superadditive channels: random unitary/orthogonal channel (Hastings '09)

Auxiliary resources for communication

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- Quantum channels: shared entanglement between the sender and receiver

Classical Capacity with entanglement assistance

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- If two parties have unlimited shared entanglement, this gives the classical capacity with entanglement assistance C_E (Bennett et al. '99,'02)

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- $C_E(\mathcal{N}) = I(\mathcal{N}) = \max_\sigma I(A;B)_\sigma$
  $\sigma_{AB} = \mathcal{N}_{A' \to B}(\phi_{AA'})$
  $I(A;B)_\sigma = S(A)_\sigma + S(B)_\sigma - S(AB)_\sigma$

Classical Capacity with (unlimited) entanglement assistance

- Single-letter! Efficiently computable! This is due to the additivity of mutual information of a channel $I(\mathcal{N}_1 \otimes \mathcal{N}_2) = I(\mathcal{N}_1) + I(\mathcal{N}_2)$
- In analogy with Shannon's capacity formula of classical channels
- Unifies classical and quantum capacity (with unlimited entanglement assistance)
- Provide ideas for channel simulation
- Entanglement assistance simplies quantum Shannon theory

Our Question

- What exactly causes C_E to be additive?
  Is it entanglement assistance?
- Since classical capacity can be additive or superadditive, how robust is the additivity property against entanglement assistance?

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- rate of entanglement consumption

$$\sum_i p_i S(\rho_i)$$

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- 1-shot classical capacity with entanglement assistance P:

$$C_P^{(1)}(\mathcal{N}) = \max_{\substack{\{p_i,\rho_i\} \\ \sum_i p_i S(\rho_i) \leq P}} \sum_i p_i S(\rho_i) + S\left(\mathcal{N}\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i S(\mathcal{N} \otimes \mathcal{I}(\phi_i))$$

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Classical Capacity with Limited Entanglement Assistance -The Additivity Problem

- Additive: $\forall P > 0, \quad C_P(\mathcal{N}) = C_P^{(1)}(\mathcal{N})$.
- Superadditive: $\exists P > 0$ s.t. $C_P(\mathcal{N}) > C_P^{(1)}(\mathcal{N})$.

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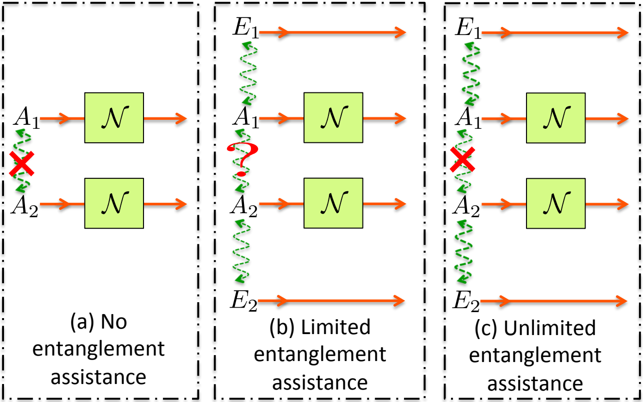

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- You may wish to use Bell pairs for a fraction of channel uses, and use pure states for the other fraction (time-sharing strategy)
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- Not always optimal!
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- Inconclusive result from unital qubit channels and entanglement-breaking channels

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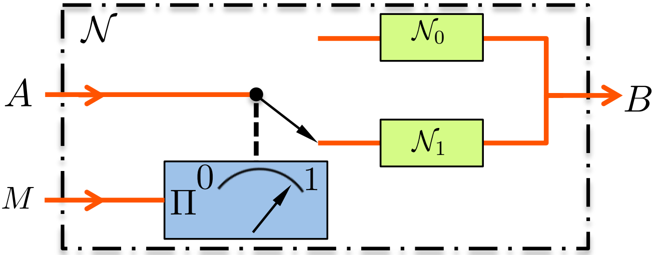
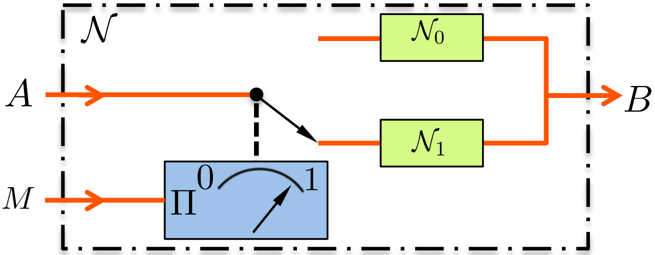
- Example via the switch channel



$$\mathcal{N}(\rho^{MA}) = \mathcal{N}_0\left(\langle 0|\rho^{MA}|0\rangle^{M}\right) + \mathcal{N}_1\left(\langle 1|\rho^{MA}|1\rangle^{M}\right)$$

Switch Channel: Intuition



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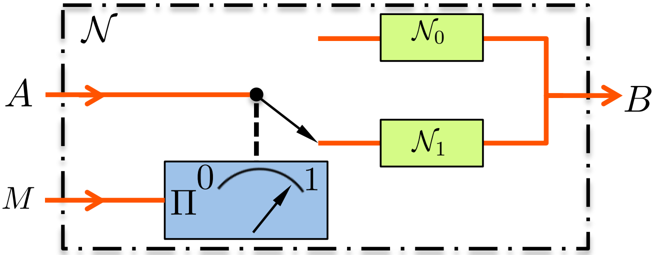

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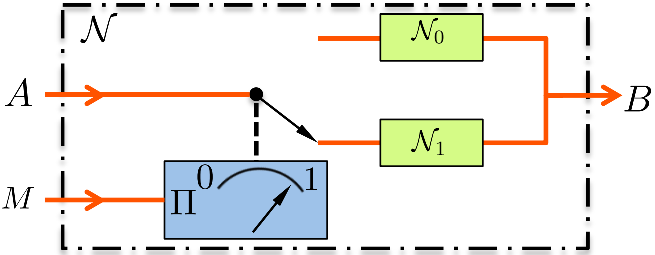

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- $P = 0$, $\mathcal{N}_0$ preferred over $\mathcal{N}_1$, $C(\mathcal{N})$ additive.
- As P increases, $\mathcal{N}_1$ gets more and more favoured. Superadditivity in $C_P(\mathcal{N}_1)$ is carried over to $C_P(\mathcal{N})$.

Switch Channel-Formalism

- Need to express $C_P^{(1)}(\mathcal{N}), C_P(\mathcal{N})$ in terms of $\mathcal{N}_0, \mathcal{N}_1$.
- Lemma: Superposition and entanglement of the measurement register is not needed to achieve $C_P^{(1)}$.
- Optimal input: $\{p_{ij}, |j\rangle\langle j|^M \otimes \rho_{ij}^A\}$, $j = 0, 1$

Switch Channel- Capacity

$$C_P^{(1)}(\mathcal{N}) = \max_{\substack{\{p_i,\rho_i\} \\ \sum_i p_i S(\rho_i) \leq P}} \sum_i p_i S(\rho_i) + S\left(\mathcal{N}\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i S(\mathcal{N} \otimes \mathcal{I}(\phi_i))$$

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- Added assumption: For $\mathcal{N}_0$ and $\mathcal{N}_1$, their capacity $C_P^{(n)}$ can be achieved with inputs such that this term is maximized.

Switch Channel - Capacity

Since $\mathcal{N}_0$ is a classical channel, with all capacities additive,

$$C^{(1)}(\mathcal{N}) = \max\left\{C\left(\mathcal{N}_0\right), C^{(1)}\left(\mathcal{N}_1\right)\right\}$$

$$C(\mathcal{N}) = \max\left\{C\left(\mathcal{N}_0\right), C\left(\mathcal{N}_1\right)\right\}$$

$$C_P^{(1)}\left(\mathcal{N}\right) = \max_{\substack{\{q,P'\} \\ (1-q)P'=P}} qC\left(\mathcal{N}_0\right) + (1-q)C_{P'}^{(1)}\left(\mathcal{N}_1\right)$$

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Switch Channel - Capacity

- With the assumption $C(\mathcal{N}_0) = C(\mathcal{N}_1)$

$$
\begin{aligned}
C_P(\mathcal{N}) &= C_P(\mathcal{N}_1) \\
C_P^{(1)}(\mathcal{N}) &= \tilde{q} C(\mathcal{N}_1) + (1-\tilde{q}) C_{\tilde{P}}^{(1)}(\mathcal{N}_1)
\end{aligned}
$$

$\tilde{q}, \tilde{P}$ achieve maximum

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- Not done yet! Consider the following chain of inequality

$$\begin{aligned} C_P\left(\mathcal{N}\right) &\geq \tilde{q} C\left(\mathcal{N}_1\right) + (1-\tilde{q}) C_{\tilde{P}}\left(\mathcal{N}_1\right) \\ &\geq \tilde{q} C\left(\mathcal{N}_1\right) + (1-\tilde{q}) C_{\tilde{P}}^{(1)}\left(\mathcal{N}_1\right) = C_P^{(1)}\left(\mathcal{N}\right) \end{aligned}$$

- Both inequalities may not be strict

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One more assumption: $C_P(\mathcal{N}_1)$ superadditive at P and beats all time-sharing strategies (strictly concave in P)

- $\tilde{P} = P$, second inequality strict (or $C_P(\mathcal{N}_1) > C_P^{(1)}(\mathcal{N}_1)$ by definition)
- $\tilde{P} > P$, first inequality strict

All assumptions for $\mathcal{N}_0$ and $\mathcal{N}_1$

- $\mathcal{N}_0$: classical, maximal output. $C(\mathcal{N}_0) = C(\mathcal{N}_1)$.
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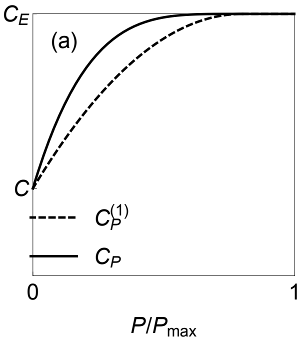
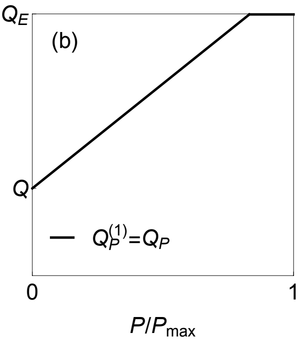
- $C_P(\mathcal{N}_1)$ strictly concave in P
- Hasting's construction plus unital extension can satisfy 1st,2nd assumption. We can modify it to satisfy 3nd assumption (technical).

Conclusion

- We constructed a channel with an additive classical capacity, but a superadditive classical capacity with limited entanglement assistance.
- What exactly gives additivity of C_E? Perhaps it is ‘unlimited‘ entanglement assistance
- Additivity is a very fragile property.





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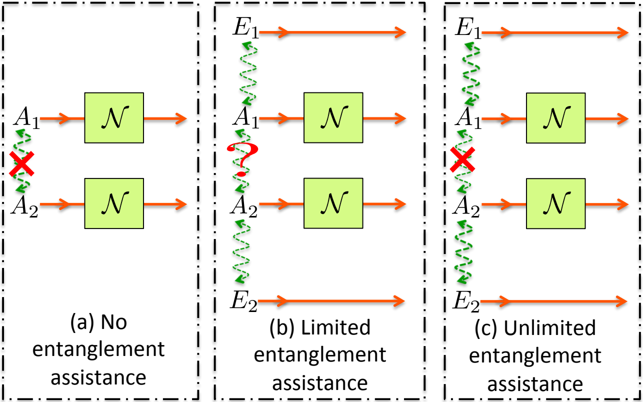

As one starts increasing entanglement assistance, one also need to increase entanglement across channel inputs in order to achieve capacity.

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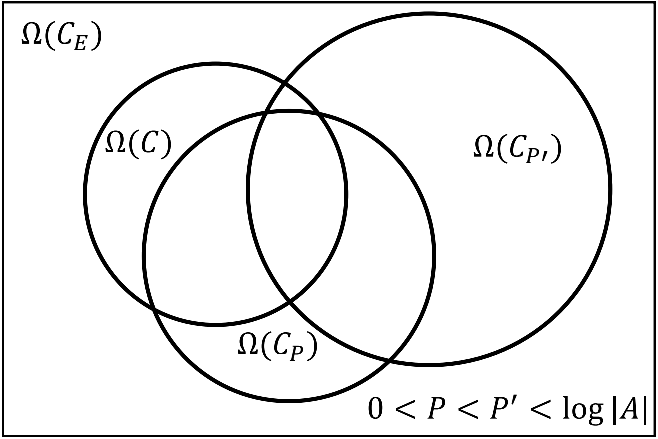

$\Omega(C_P)$: all quantum channels with C_P additive

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- One can also look at it from the perspective of trade-off capacities
- C additive, Q additive, CQ trade-off can be superadditive. Many other results in CQE trade-off.
- arXiv: 1708.04314
  with Quntao Zhuang, Min-Hsiu Hsieh, Peter Shor
- Thank you!