The quantum monad on relational structures


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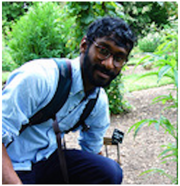
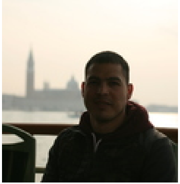

Nadish de Silva[2]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1. Introduction

Motivation

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- use **quantum resources** for information-processing tasks
- delineate the scope of **quantum advantage**

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- A setting in which this has been explored is **non-local games**

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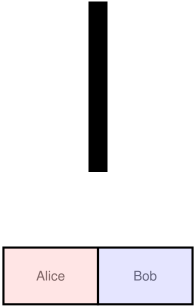
Alice and Bob cooperate in solving a task set by Verifier

May share prior information,



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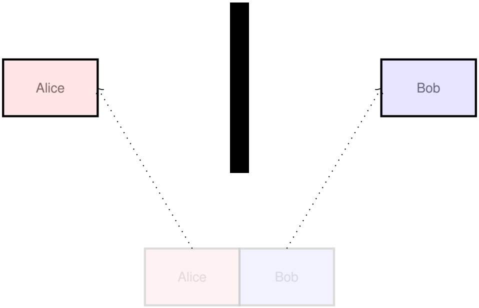

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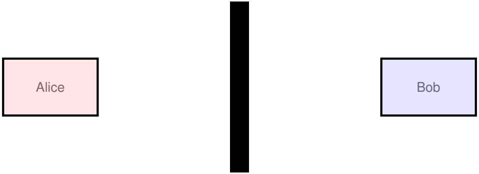

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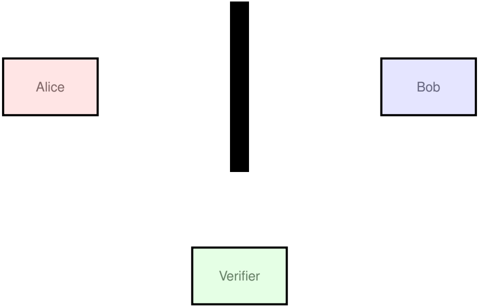

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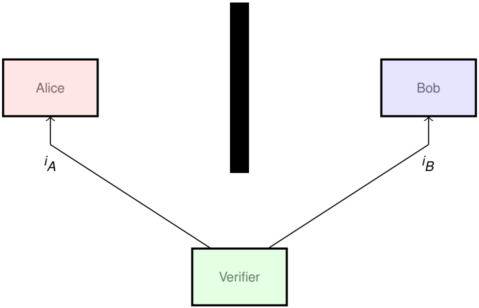

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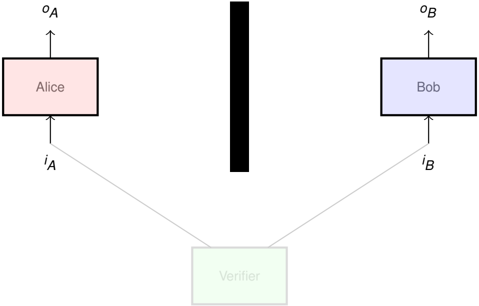

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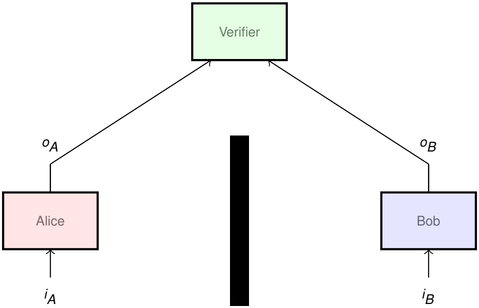

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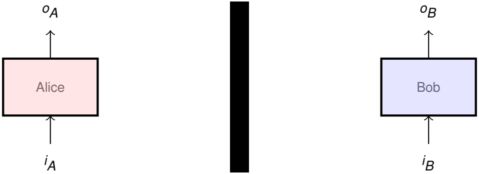

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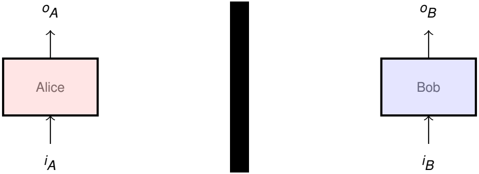

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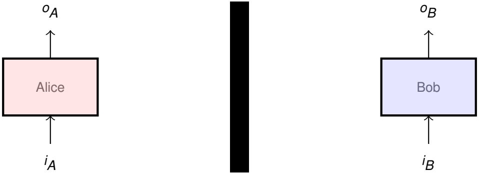

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A **perfect strategy** is one that wins with probability 1.

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  Game has (classical) perfect strategy iff there is a solution.
  E.g.: the system of linear equations over $\mathbb{Z}_2$

$$
\begin{array}{lcl}
A \oplus B \oplus C = 0 & \qquad & A \oplus D \oplus G = 0 \\
D \oplus E \oplus F = 0 & \qquad & B \oplus E \oplus H = 0 \\
G \oplus H \oplus I = 0 & \qquad & C \oplus F \oplus I = 1
\end{array}
$$

  has a quantum solution but no classical solution.

Examples of non-local games

- Cleve, Mittal, Liu, Slofstra:
  games for binary constraint systems $\rightsquigarrow$ quantum solutions

- Cameron, Montanaro, Newman, Severini, Winter:
  game for graph colouring $\rightsquigarrow$ quantum chromatic number

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Many of these works have some aspects in common. We aim to flesh this out by subsuming them under a common framework.

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Finite relational structures and homomorphisms

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- constraint satisfaction
- finite model theory
- theory of relational databases
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Many relevant questions in these areas can be phrased in terms of (existence, number of, . . .) homomorphisms between finite relational structures.

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- We formulate the task of constructing a homomorphism between two relational structures as a **non-local game**
- **uniformly** obtain quantum analogues *for free* for a whole range of classical notions from CS, logic, . . .

Outline of the talk

- Introduce homomorphism game for relational structures
- Arrive at the notion of quantum homomorphism,
  which removes the two-player aspect of the game

  (generalises Cleve & Mittal and Mančinska & Roberson)
- Quantum monad: capture quantum homomorphisms as classical homomorphisms to a *quantised* version of a relational structure

  (inspired on Mančinska & Roberson for graphs)
- Connection between non-locality and state-independent strong contextuality

2. Homomorphisms game for relational structures

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

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A σ-**structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

- A is a non-empty set,
- for each $l \in [p]$, $R_l^{\mathcal{A}} \subseteq A^{k_l}$ is a relation of arity k_l on A.

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A homomorphism of σ-structures $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f : A \longrightarrow B$ such that for all $l \in [p]$ and $\mathbf{x} \in A^{k_l}$,

$$\mathbf{x} \in R_l^{\mathcal{A}} \quad \Longrightarrow \quad f(\mathbf{x}) \in R_l^{\mathcal{B}}$$

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(For simplicity, from now on consider a single relational symbol R of arity k)

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What about quantum resources?

Homomorphism game with quantum resources

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These resources are used as follows:

- Given input $\mathbf{x} \in R^{\mathcal{A}}$, Alice measures $\mathcal{E}_{\mathbf{x}}$ on her part of ψ
- Given input $x \in A$, Bob measures $\mathcal{F}_x$ on his part of ψ
- Both output their respective measurement outcomes
- $P(\mathbf{y}, y \mid \mathbf{x}, x) = \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi$

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Perfect strategy conditions:

$$
\begin{array}{lll}
\text{(QS1)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \quad I \quad)\psi = 0 & \text{if } \mathbf{y} \notin R^{\mathcal{B}} \\
\text{(QS2)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0 & \text{if } x = \mathbf{x}_i \text{ and } y \neq \mathbf{y}_i
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3. From quantum perfect strategies to quantum homomorphisms

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

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Simplifying quantum strategies

The reduction proceeds in three steps:

1. The state and strategies are projected down to the support of the Schmidt decomposition of the state. This reduces the dimension of the Hilbert space and preserves the probabilities of the strategy exactly.

2. It is shown that this strategy must already satisfy strong properties (PVMs and $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{\mathbf{x}_i,y}$).

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N.B. In passing to the special form, the dimension is **reduced**; the process by which we obtain projective measurements is not at all akin to dilation.

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Theorem The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

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These $P_{x,y}$ are enough to determine the strategy!

Quantum homomorphisms

A quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in some dimension $d \in \mathbb{N}$ satisfying:

(QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} = I$.

(QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$,

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so we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

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Theorem For finite structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. The ($\mathcal{A}$,$\mathcal{B}$)-homomorphism game has a quantum perfect strategy.
2. There is a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{A} \xrightarrow{q} \mathcal{B}$)

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
- (classical) homomorphisms from $\mathcal{A}$ to $\mathcal{Q}_d\mathcal{B}$

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Underlying set of the structure $\mathcal{Q}_d\mathcal{A}$ is the set of d-dimensional projector-valued distributions on A.

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Quantum homomorphisms as Kleisli maps

$\mathcal{Q}_d$ is part of a **graded monad** on the category of relational structures and (classical) homomorphisms.

Monads play a major rôle in programming language theory, providing a uniform way of encapsulating various notions of computation:

- partiality
- exceptions
- non-determinism
- probabilistic
- state updates
- input/output
- ...

$\rightsquigarrow$ composition of quantum homomorphisms, keeping track of the dimension of the resources

Composition of perfect strategies

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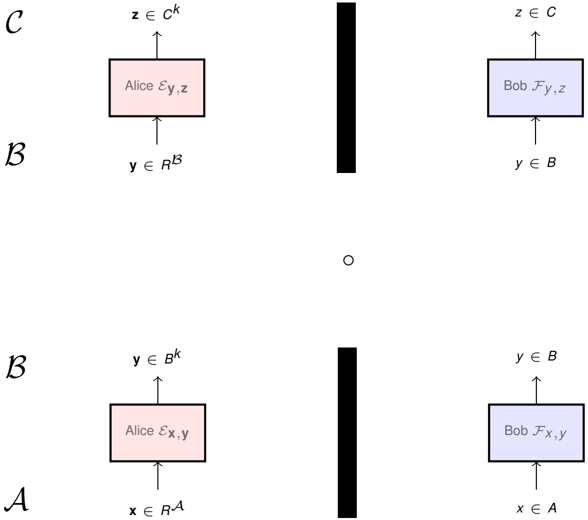

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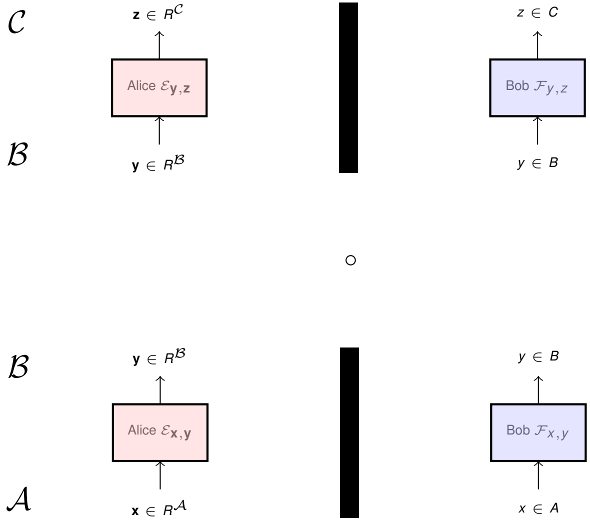

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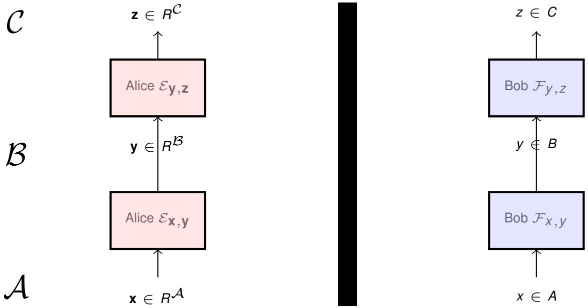

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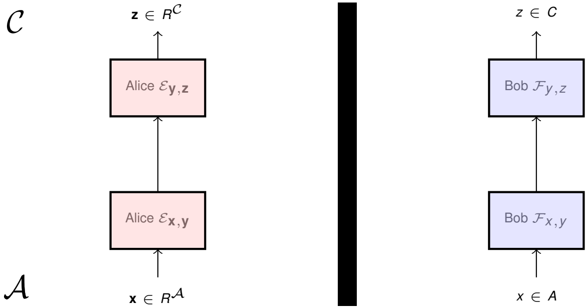
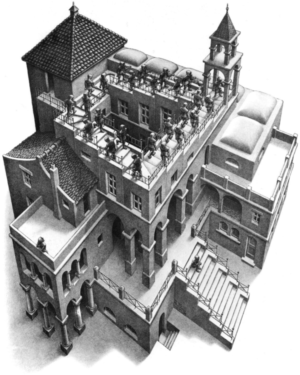


4. Contextuality and non-locality

Contextuality

Contextuality is a fundamental feature of quantum mechanics, which distinguishes it from classical physical theories.

It can be thought as saying that empirical predictions are inconsistent with all measurements having pre-determined outcomes.



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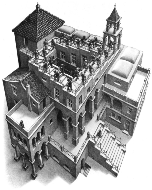
Non-locality is a particular case of contextuality for Bell scenarios

…but here we show that certain contextuality proofs can always be underwritten by non-locality arguments.



Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
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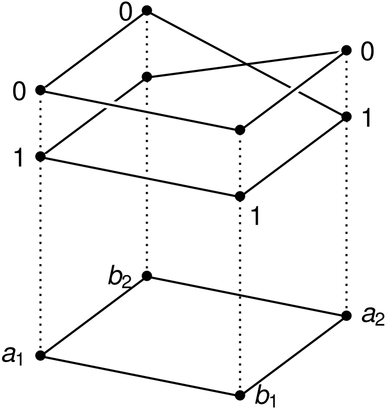
Strong contextuality: if there is no global assignment $g : X \longrightarrow O$ such that for all $C \in \mathcal{M}$, $e_C(g|_C) = 1$. That is, no global assignment is consistent with the model in the sense of yielding **possible** outcomes in all contexts.

E.g.: GHZ, Kochen–Specker, (post-quantum) PR box

Strong contextuality

Strong Contextuality:
no consistent global assignment.

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
a_2	b_2	×	✓	✓	×



Strong contextuality and constraint satisfaction

The support of e can be described as a CSP $\mathcal{K}_e$

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- solutions for $\mathcal{K}_e$
- (homomorphisms $\mathcal{A}_{\mathcal{K}_e} \longrightarrow \mathcal{B}_{\mathcal{K}_e}$)
- consistent global assignements for *e*

$\boxed{\text{Hence, } e \text{ is strongly contextual iff } \mathcal{K}_e \text{ has no (classical) solution.}}$

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
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> Quantum state-independent strong contextuality:
> $\mathcal{K}_e$ has quantum solution but no classical solution!

General way of turning state-independent contextuality proofs into Bell non-locality arguments (generalising Heywood & Redhead's construction).

5. Outlook

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- What about state-dependent strong contextuality?
- A strategy has a winning probability. Can we adapt this to deal with quantitative aspects (contextual fraction, ...)?
- Homomorphisms are related to the existential positive fragment: can this be extended to provide quantum vality for first-order formulae?
- Also, quantum versions of pebble games
  $\rightsquigarrow$ quantum finite model theory

Thank you!

Questions...

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