Generalized entanglement entropies of quantum designs

Zi-Wen Liu

Center for Theoretical Physics
Department of Physics
MIT

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Summarizes the QI results in 1703.08104 (scrambling complexity)
Joint work with: Seth Lloyd, Elton Zhu, Huangjun Zhu

Outline

Introduction

Entanglement properties of random states/dynamics

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- Play key roles in: scrambling, quantum chaos, many-body localization, quantum error correction, quantum crypto...

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⋆ This work: higher order entanglement entropies vs. pseudorandom states/unitary channels

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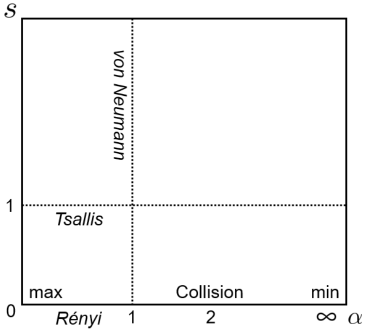

$s = 1$: Tsallis
$s \to 0$: Rényi ($\alpha \to \infty$: min)

$\alpha \to 1$: von Neumann

Rényi entropies

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$\alpha \to \infty$ limit: **min entropy**

$$S_{\min}(\rho) = -\log \|\rho\| = -\log \lambda_{\max}(\rho).$$

Determined only by the op. norm/largest eigenvalue.
Strongest entropy: large only when the whole spectrum is almost uniform.

Rényi entropies

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Results for other families (eg Tsallis) are also obtained.

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$$\boldsymbol{\lambda} = \Big(\frac{1}{\sqrt{d}}, \underbrace{\frac{1-\frac{1}{\sqrt{d}}}{d-1}, \cdots, \frac{1-\frac{1}{\sqrt{d}}}{d-1}}_{d-1}\Big).$$

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- Rényi-2 (and thus vN): almost maximal

$$S_R^{(2)}(\boldsymbol{\lambda}) \geq \log d - 1.$$

Designs

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Finite-order approximation to the Haar measure (pseudorandom).

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- **State design**: An ensemble/dist. ν of pure state vectors in dimension d is a (complex projective) t-*design* if

$$\mathbb{E}_\nu\, p(\psi) = \int \mathrm{d}\psi p(\psi) \quad \forall p \in \mathrm{Hom}_{(t,t)}(\mathbb{C}^d).$$

- **Unitary design**: An ensemble/dist. μ of unitary operators in dimension d is a *unitary* t-*design* if

$$\mathbb{E}_\mu\, p(U) = \int \mathrm{d}U p(U) \quad \forall p \in \mathrm{Hom}_{(t,t)}(\mathrm{U}(d)).$$

Integrals taken over: uniform measure on the complex unit sphere in $\mathbb{C}^d$/Haar measure on $\mathrm{U}(d)$.
$\mathrm{Hom}_{(t,t)}$: polynomials homogeneous of degree t both in the elements and in their complex conjugates.

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Efficient to implement. Applications in: signal processing, randomized benchmarking, quantum data hiding, decoupling...

Results for random states

Consider a bipartite pure state $|\psi\rangle_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimensions d_A and d_B respectively.

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Entanglement entropy: entropy of the reduced density operator $\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$.

Theorem: equal partition, asymptotic

Let ν_α be a projective α-design. Consider equal partitions $d_A = d_B$. As $d_A \to \infty$,

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \geq \log d_A - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} + O(d_A^{-2}).$$

$\mathrm{Cat}_\alpha := \frac{1}{\alpha+1}\binom{2\alpha}{\alpha}$: Catalan number. So,

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- Rényi-α entanglement entropy averaged over an α-design (expectation) is almost maximal.
- A state sampled from an α-design is very likely to exhibit almost maximal Rényi-α entanglement entropy.

Results for random states

Order correspondence

Theorem: general partition, finite dimension

Let ν_α be a projective α-design. Let $q := \alpha^3/(32d_B^2) < 1, h(q) := 1 + 2q/[3(1-q)]$. For all $d_A, d_B, 0 \leq \alpha \leq \infty$,

$$\begin{aligned}\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) &\geq \log d_A - \frac{2\alpha - \frac{3}{2}\log\alpha + \log h(q) - \frac{1}{2}\log\pi}{\alpha - 1} \\ &\geq \log d_A - 2.\end{aligned}$$

When $d_A < d_B$, the result can be improved as follows:

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \geq \log d_A - 2\sqrt{\frac{d_A}{d_B}} - \log c,$$

where $c = 1$ if $\mathcal{H}$ is real and $c = 2$ if $\mathcal{H}$ is complex.

Methods:

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- Key observations: given an α-design ν_α, $\mathbb{E}_{\nu_\alpha} \mathrm{tr}\{\rho_A^\alpha\} = \int \mathrm{d}\psi \mathrm{tr}\{\rho_A^\alpha\}$, since $\mathrm{tr}\{\rho_A^\alpha\}$ only involves degree-α terms of entries of $|\psi\rangle$.

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  By the convexity of Rényi in $\operatorname{tr}\{\rho^\alpha\}$ and Jensen's ineq.,

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}\psi \operatorname{tr}\{\rho_A^\alpha\}\right).$$

- Boils down to calculating the Haar integrals of $\mathrm{tr}\{\rho_A^\alpha\}$:

$$\int \mathrm{d}\psi \mathrm{tr}\{\rho_A^\alpha\} = \frac{1}{\alpha! D_{[\alpha]}} \sum_{\sigma \in S_\alpha} d_A^{\xi(\sigma\tau)} d_B^{\xi(\sigma)}.$$

$D_{[\alpha]} := \binom{d_A d_B + \alpha - 1}{\alpha}$: dim of the symmetric subspace of $\mathcal{H}^{\otimes \alpha}$
S_α: symmetric group of α symbols
$\xi(\sigma)$: number of disjoint cycles of permutation σ
τ: full cycle/1-shift $(1\ 2\ \ldots\ \alpha)$
Similar results Zyczkowsky-Sommers '01; Collins-Nechita '10, '11

- Equal partitions, large d limit (basic result):
  Cycle Lemma. $\xi(\sigma\tau) + \xi(\sigma) \leq \alpha + 1$.
  After a little algebra:

$$\int \mathrm{d}\psi \mathrm{tr}\{\rho_A^\alpha\} = \mathrm{Cat}_\alpha d_A^{-\alpha+1} + O\left(d_A^{-(\alpha+1)}\right).$$

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- General partitions, finite dimension (general result): more technical.
  Used various tools developed in random matrix theory, Weingarten calculus and representation theory etc.
  Collins-Matsumoto '17; Goupil-Schaeffer '98...
  See full paper for details.

The above results are for exact designs. The following error bound shows that these results does not deviate much for approximate designs:

Theorem

Let $\tilde{\nu}_\alpha$ be an ϵ-approximate α-design ($\alpha \geq 2$), i.e., $\left\|\mathcal{F}_\alpha(\tilde{\nu}_\alpha) - P_{[\alpha]}\right\|_1 \leq \epsilon$, where $\mathcal{F}_\alpha(\tilde{\nu}_\alpha) := D_{[\alpha]}\, \mathbb{E}_{\tilde{\nu}_\alpha}(|\psi\rangle\langle\psi|)^{\otimes t}$ is the α-th frame operator of $\tilde{\nu}_\alpha$, and $P_{[\alpha]}$ is the projector onto the α-partite symmetric subspace of $(\mathbb{C}^d)^{\otimes\alpha}$. Then

$$\mathbb{E}_{\tilde{\nu}_\alpha} S_R^{(\alpha)}(\rho_A) \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}\psi\, \mathrm{tr}\{\rho_A^\alpha\} + \frac{\epsilon}{D_{[\alpha]}}\right).$$

Key step: matrix Hölder

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Can finite order designs maximize ∞-entropy (min entropy)?
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Let ν_α be a projective α-design, where
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$$\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_A) \geq \log d_A - 2 - a.$$

In particular, $\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_A) \geq \log d_A - 3$ if $\alpha = \lceil \log d_A \rceil$.

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Key step: $\mathbb{E}_{\nu_\alpha} \|\rho_A\| \leq (\mathbb{E}_{\nu_\alpha} \|\rho_A\|^\alpha)^{1/\alpha} \leq [\mathbb{E}_{\nu_\alpha} \operatorname{tr}\{\rho_A^\alpha\}]^{1/\alpha}$.

Another natural question: Are Rényi entanglement entropies of different orders truly separated, in the sense that $\exists$ α-design s.t. Rényi entropy of order $> \alpha$ is bounded away from maximal?

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We construct such a separation for $\alpha = 2$:

Theorem

There exist a family of 2-designs such that, for all $\alpha > 2$, $\log d_A - S_R^{(\alpha)}(\rho_A) \in \Omega(\log d_A)$.

Our construction is based on the orbits of a special subgroup of the unitary group on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Representation theory.

Results for random states

Separation: existence of 2-design with non-maximal Rényi-3

Let $G = \mathrm{U}_A \otimes \mathrm{U}_B$ (NB: irreducible, not unitary 2-design). The orbit of $|\psi\rangle$ under the action of G forms a 2-design iff $\mathrm{tr}\{\rho_A^2\}$ is equal to the average over the uniform ensemble:

$$\mathrm{tr}\{\rho_A^2\} = \frac{d_A + d_B}{d_A d_B + 1}.$$

It holds if ρ_A has the following spectrum

$$\lambda_1 = \frac{d_A d_B + 1 + (d_A - 1)\sqrt{(d_A + 1)(d_A d_B + 1)}}{d_A(d_A d_B + 1)},$$

$$\lambda_2 = \cdots = \lambda_{d_A} = \frac{d_A d_B + 1 - \sqrt{(d_A + 1)(d_A d_B + 1)}}{d_A(d_A d_B + 1)}.$$

Suppose d_B/d_A is bounded by constant r. Then $\lambda_1 \geq (r d_A)^{-1/2}$. So

$$S_R^{(\alpha)}(\rho_A) \leq \frac{1}{1-\alpha}\lambda_1^{\alpha} \leq \frac{\alpha}{2(\alpha - 1)}(\log d_A + \log r).$$

Results for random unitaries

Model: entanglement/tripartite information of the Choi state

Intrinsic entanglement/scrambling properties of random unitary channels?

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Choi isomorphism:

Unitary operator $U = \sum_{i,j=0}^{d-1} U_{ij}|i\rangle\langle j|$

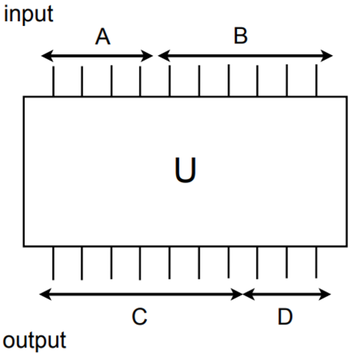
$$\Updownarrow$$

Pure state $|U\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} U_{ji}|i\rangle_{in} \otimes |j\rangle_{out}$

Results for random unitaries

Model: entanglement/tripartite information of the Choi state

Partition the input into A and B, and the output into C and D



We are interested in the entanglement between AC and BD (entropy of AC).

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Model: entanglement/tripartite information of the Choi state

(Negative) tripartite information

$$-I_3(A:C:D) := I(A:CD) - I(A:C) - I(A:D).$$

Determined by $S(AC)$.

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  Large $-I_3$ diagnoses 'scrambling' Hosur-Qi-Roberts-Yoshida '15: local operators supported on A always gets spread onto CD via global entanglement.

Results for random unitaries

Order correspondence

Theorem: equal partition, asymptotic

Let μ_α be a unitary α-design. Consider equal partitions of the input and output registers, $d_A = d_B = d_C = d_D$. As $d \to \infty$,

$$\mathbb{E}_{\mu_\alpha} S_R^{(\alpha)}(\rho_{AC}) \geq \log d - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} + O(d^{-1}).$$

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- Rényi-α entanglement entropy averaged over a unitary α-design (expectation) is almost maximal.
- A unitary sampled from a unitary α-design is very likely to exhibit almost maximal Rényi-α entanglement entropy.

Theorem: general partition, finite dimension

Let μ_α be a unitary α-design. Suppose $d > \sqrt{6}\alpha^{7/4}, d_A \leq d_B$. Then

$$\begin{aligned} & \mathbb{E}_{\mu_\alpha} S_R^{(\alpha)}(\rho_{AC}) \\ \geq \quad & \log d - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} - \frac{\log\left[\frac{a_\alpha h(q)}{8}\left(7 + \cosh\frac{2\alpha(\alpha-1)}{d}\right)\right]}{\alpha - 1}, \end{aligned}$$

where $a_\alpha := \left(1 - \frac{6\alpha^{7/2}}{d^2}\right)^{-1}$.

Methods:

Results for random unitaries

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<u>Methods</u>:

- By properties of designs and Jensen (similar as for states):

$$\mathbb{E}_{\nu_\alpha}\left[S_R^{(\alpha)}\left(\rho_{AC}\right)\right] \geq \frac{1}{1-\alpha}\log\left(\int \mathrm{d}U \operatorname{tr}\left\{\rho_{AC}^{\alpha}\right\}\right).$$

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- Haar integrals of $\mathrm{tr}\{\rho_A^\alpha\}$:

$$\int \mathrm{d}U \mathrm{tr}\left\{\rho_{AC}^\alpha\right\} = \frac{1}{d^\alpha}\sum_{\sigma,\gamma\in S_\alpha} d_A^{\xi(\sigma\tau)} d_B^{\xi(\sigma)} d_C^{\xi(\gamma\tau)} d_D^{\xi(\gamma)} \mathrm{Wg}(d,\sigma\gamma^{-1}),$$

$\mathrm{Wg}(d,\sigma) := \frac{1}{(\alpha!)^2}\sum_{\lambda\vdash\alpha}\frac{\chi^\lambda(1)^2\chi^\lambda(\sigma)}{s_{\lambda,d}(1,\cdots,1)}$: Weingarten function.
$\lambda\vdash\alpha$ means λ is a partition of α
χ^λ, s_λ: corresponding character of S_α and Schur polynomial
Wg can be derived by various tools in representation theory, such as Schur-Weyl duality and Jucys-Murphy elements.

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  Collins '03; Collins-Sniady '06;

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- Equal partitions, large d limit (basic result):
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  Collins '03; Collins-Sniady '06;
- General partitions, finite dimension (general result):
  See full paper for details.

- Equal partitions, large d limit (basic result):
  the Cycle Lemma + asymptotics of Weingarten function
  Collins '03; Collins-Sniady '06;
- General partitions, finite dimension (general result):
  See full paper for details.
- Results for unitaries analogous to those for states (analysis much more difficult). Gap constructions unknown.

Error bound for approximate unitary designs:

Theorem

Let $\tilde{\mu}_\alpha$ be an ϵ-approximate unitary α-design, i.e., $\left\| \mathcal{F}_\alpha(\tilde{\mu}_\alpha) - \int \mathrm{d}U U^{\otimes\alpha} \otimes U^{\dagger\otimes\alpha} \right\|_1 \leq \epsilon$, where $\mathcal{F}_\alpha(\tilde{\mu}_\alpha) := \mathbb{E}_{\tilde{\mu}_\alpha}\left[U^{\otimes\alpha} \otimes U^{\dagger\otimes\alpha} \right]$ is the α-th frame operator of $\tilde{\mu}_\alpha$.
Then

$$\mathbb{E}_{\tilde{\mu}_\alpha}[S_R^{(\alpha)}(\rho_{AC})] \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}U \operatorname{tr}\{\rho_{AC}^\alpha\} + \frac{1}{d^\alpha}\epsilon \right).$$

Key step: matrix Hölder

Similar techniques and results for the min entropy:

Theorem

Let μ_α be a unitary α-design, where $1 \leq \alpha = \lceil \log d/a \rceil \leq \sqrt{d}/2$ and $a > 0$; then

$$\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_{AC}) \geq \log d - 2 - a.$$

In particular, $\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_{AC}) \geq \log d - 3$ if $\alpha \geq \lceil \log d \rceil$.

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Cannot distinguish higher (superlog) order unitary designs from the Haar measure by entanglement properties.

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- Linking the order of design and generalized entropy: α-design exhibits almost maximal entanglement as measured by Rényi-α.
  A family of strong Page's theorems.
- Logarithmic 'nontrivial' orders of design: log-design maximizes all entanglement entropy.

Entropic scrambling complexities

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  Fast max-scrambling conjecture: the minimum time for a physical system to max-scramble scales as $\tilde{O}(n)$.
  This afternoon Nakata-Hirche-Koashi-Winter '17

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- Open problems: Separations for higher orders and the unitary case? Negative tripartite Rényi information (no subaddivity)?
- Dynamical aspects of designs/complexity?
  eg Design Hamiltonian Nakata-Hirche-Koashi-Winter '17
- Applications in the study of quantum gravity, quantum many-body physics, quantum cryptography...

Thanks for your attention!