No-Hypersignaling Principle


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Motivation

General probabilistic theories (GPTs) provide a mathematical description of Nature in terms of states, effects, and a rule to compose them in order to obtain observable correlations.

Among GPTs, the standard formalization of quantum theory comprises a set of merely mathematical axioms (the Hilbert space formalism), with no direct operational interpretation.

The motivation of the program of GPTs is to single out quantum theory on the basis of principles that are:

- operational, rather than just mathematical;
- optionally, device-independent (DI).

Definition (Device-independent principle)

A principle is DI iff for any theory that violates such a principle, the violation can be detected by a DI test, i.e. a test based on an observed correlation and that only assumes the causal relations among events.
Causal relations and device-independence

- A and B are **space-like** separated, can only share space-like correlations $\Omega$, no-signaling applies;
- A and $A'$ are **time-like** separated, information can be encoded into $\{\Omega_x\}$ and decoded by $\{E_y\}$, no-signaling does not apply.

Principles only constraining correlations among **space-like** separated events are **DI**, since the no-signaling principle allows any violation to be detected by a DI test, as is the case for local-realism and Bell test:
Principles simultaneously constraining **space- and time-like** correlations cannot be DI, since no-signaling does not apply.

E.g. **information causality** (IC) constraints correlations $p(b|x, y)$ obtained by exchanging a limited amount of classical information $f(x, a)$, an assumption that cannot be certified in a DI way:

$$\phi$$

$x \xrightarrow{\pi_{a|x}} f(x, a) \xrightarrow{\pi_{b|y,f(x,a)}} b$

However, a **sufficient condition** for IC violation can be obtained in a DI way by performing a _purely space-like_ test, i.e. a Bell test:

$$\phi$$

$x \xrightarrow{\pi_{a|x}} a \quad y \xrightarrow{\pi_{b|y}} b$

After $p(a, b|x, y)$ is collected, one checks if there exists a postprocessing $y, b, f(x, a) \rightarrow b'$ such that $p(b'|x, y)$ violates IC.
Motivation: constraining space-like correlations or space- and time-like correlations does not single out quantum theory: we push these ideas further by constraining purely time-like correlations.

Our general program: to derive a theoretical framework to characterize the time-like correlations compatible with any GPT.

Applications: we fully characterize all the bipartite extension of a sqit. We show that one such extension is such that:

- does not contradict classical and quantum theories at the level of space-like correlations,
- displays an anomalous behavior in its time-like correlations.

We formalize this anomaly in terms of a novel principle, that we call no-hypersignaling, that inherently constrains time-like correlations, hence its violations cannot be detected in a DI way.
We consider the **purely time-like** setup of a memory in which, upon input of $x$, Alice prepares system $S$ into state $\Omega_x$ and transmits it to herself in the future:

$$x \xrightarrow{\Omega_x} S \xrightarrow{E_y} y$$

Let $\mathcal{P}_{S}^{m\rightarrow n}$ denote the convex hull generated by all $m$-input/$n$-output conditional probability distributions $p_{y|x}$ that can be obtained by transmitting one elementary system $S$.

We denote with $C_d$ and $Q_d$ a $d$-dimensional classical system and a quantum system with $d$-dimensional Hilbert space, respectively.
Frenkel and Weiner’s theorem

Frenkel and Weiner recently proved this remarkable result\(^1\):

\[ p_{C_d}^{m\to n} = p_{Q_d}^{m\to n}, \quad \forall m, n. \]

Comparison with the **Holevo bound**: the Holevo bound states that the mutual information attainable by exchanging a \( d \)-dimensional classical system and a quantum system with Hilbert-space dimension \( d \) are the same.

The Holevo bound is a statement about a *specific function* (mutual information), while Frenkel-Weiner’s theorem is about the set of attainable correlations.

Hence, the Holevo bound follows from Frenkel-Weiner’s theorem as an immediate **corollary**!

This motivates us to provide the following operational definition of dimension:

**Definition (Signaling dimension)**

The signaling dimension of a system $S$, denoted by $\kappa(S)$, is defined as the smallest integer $d$ such that $\mathcal{P}_{S}^{m \rightarrow n} \subseteq \mathcal{P}_{C_d}^{m \rightarrow n}$, $\forall m, n$.

**Properties** of the signaling dimension:

- by definition, $\kappa(S)$ equals the usual classical dimension,
- $\kappa(S)$ also equals $^2$ the usual quantum dimension, thus for brevity $\mathcal{P}_{d}^{m \rightarrow n} := \mathcal{P}_{C_d}^{m \rightarrow n} = \mathcal{P}_{Q_d}^{m \rightarrow n}$,
- $\kappa(S)$ does not depend on an arbitrarily made choice of a specific protocol (such as perfect state discrimination);
- $\kappa(S)$ is non-trivial even for those theories where perfectly discriminable states do not exist, i.e. when the no-restriction hypothesis is relaxed.

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No-hypersignaling principle

Analogously, we introduce the following for the signaling dimension:

Definition (No-hypersignaling principle)

A theory is non-hypersignaling iff, for any set of systems \( \{S_k\} \) with signaling dimensions \( \kappa(S_k) \), the signaling dimension of the composite system \( \otimes_k S_k \) satisfies

\[
\kappa(\otimes_k S_k) \leq \prod_k \kappa(S_k).
\]

Informally, in terms of input-output correlations, it must not matter if the systems are transmitted separately or jointly.

For two identical systems, no-hypersignaling becomes:

\[
P_S^{m\rightarrow n} \subseteq P_d^{m\rightarrow n} \quad \Rightarrow \quad P_{S \otimes 2}^{m\rightarrow n} \subseteq P_{d^2}^{m\rightarrow n},
\]
An example of a **hypersignaling theory**: while system $S$ satisfies $\mathcal{P}_S^m \rightarrow n \subseteq \mathcal{P}_d^m \rightarrow n$, and thus has signaling dimension $d$, the composite system $S \otimes S$ has a signaling dimension strictly larger than $d^2$.

Informally, by transmitting the systems jointly rather than separately, better correlations are achieved.

Hence, by the hyperplane separation theorem, any violation of the no-hypersignaling principle can be detected by a **game** $g$:

$$g^T \cdot p > \max_{q \in \mathcal{P}_K^m \rightarrow n} g^T \cdot q, \quad K = \prod_k \kappa(S_k).$$
A toy model theory

We start from the **sqit**, the elementary system of the theory commonly considered to produce PR correlations:

- four extremal **states** $\{\omega_x\}$ (yellow square),
- four extremal **effects** $\{e_y\}$, plus the null and unit effects $0, \bar{e}$ (blue cone).

For a sqit $S$ one has that $\mathcal{P}_{S}^{m\to n} = \mathcal{P}_{2}^{m\to n}$, that is any correlation $p_y|x$ achievable by transferring a sqit:

\[ x = \omega_x \xrightarrow{S} e_y \rightarrow y \]

is also achievable by transferring a classical bit.
All bipartite extensions of a sqit can be given in terms of:

- 24 extremal bipartite states, 8 of which are entangled,
- 24 extremal bipartite effects, 8 of which are entangled.

We derived all the self-consistent bipartite extensions of a sqit:

1. **PR model:** All the 24 states; only the 16 factorized effects;
2. **HS model:** Only the 16 factorized states; all the 24 effects;
3. **Hybrid models:** Two entangled states and effects included;
4. **Frozen Models:** Only one entangled state and effect included, but no allowed reversible dynamics.

Since, for PR model, extremal measurements have four effects, $\mathcal{P}_{S \otimes S}^{m \rightarrow n} = \mathcal{P}_{4}^{m \rightarrow n}$, i.e. PR model cannot violate no-hypersignaling.

Analogously, since HS model has no entangled states, it cannot exhibit superclassical space-like correlations.
Does the HS model violate the no-hypersignaling principle?

Consider payoff $g$ and input/output correlation $p$, achievable by transmitting a family of seven states $\{\Omega_x\}$ and performing a measurement with seven effects $\{E_y\}$ of HS model:

$$g = \frac{1}{21} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \end{pmatrix}, \quad p = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

By exchanging two classical bits, the optimal payoff is $10/21$, but $g \cdot p = 1/2 > 10/21$, thus no-hypersignaling is violated!
Conclusion

Is no-hypersignaling independent of other principles?

No-Hypersignaling vs Information Causality:
- **CT** classical theory,
- **QT** quantum theory,
- **PR** PR-boxes theory,
- **HS** hypersignaling theory.

No-Hypersignaling vs Local Tomography:
- **RQT** real quantum theory,
- **FQT** fermionic quantum theory.
Motivation: characterizing time-like correlations allowed by QT.

Our general program\(^3\):

- We introduced signaling dimension as an *operational*, *task-independent* dimension for any GPT.
- We introduced the no-hypersignaling principle as a *scaling rule* for signaling dimension under system composition.
- We derived a general *theoretical framework* to detect no-hypersignaling violations.

Applications:

- We fully characterized *all* the bipartite extensions of a squit.
- By applying our framework, we showed that the HS model’s time-like correlations violate the no-hypersignaling principle, but its space-like correlations are compatible with CT and QT.