No-Hypersignaling Principle

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Motivation

General probabilistic theories (GPTs) provide a mathematical description of Nature in terms of states, effects, and a rule to compose them in order to obtain observable correlations.

Among GPTs, the standard formalization of **quantum theory** comprises a set of **merely mathematical axioms** (the Hilbert space formalism), with no direct operational interpretation.

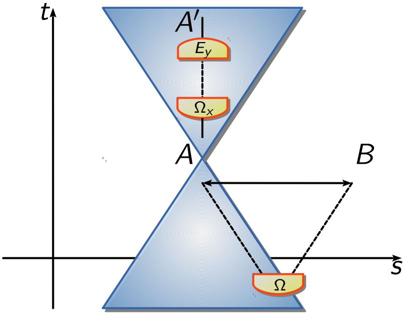
The **motivation** of the program of GPTs is to single out quantum theory on the basis of principles that are:

- **operational**, rather than just mathematical;
- **optionally, device-independent (DI)**.

Definition (Device-independent principle)

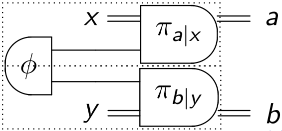
A principle is DI iff for **any** *theory that violates such a principle, the violation can be detected by a DI test, i.e. a test based on an observed correlation and that only assumes the causal relations among events.*

Causal relations and device-independence



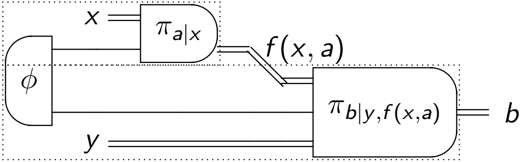
- A and B are **space-like** separated, can only share space-like correlations Ω, no-signaling applies;
- A and A' are **time-like** separated, information can be encoded into $\{\Omega_x\}$ and decoded by $\{E_y\}$, no-signaling does not apply.

Principles only constraining correlations among **space-like** separated events are **DI**, since the no-signaling principle allows any violation to be detected by a DI test, as is the case for local-realism an Bell test:

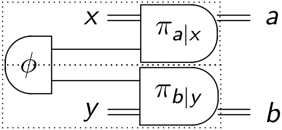


Principles simultaneously constraining **space- and time-like** correlations **cannot be DI**, since no-signaling does not apply.

E.g. **information causality** (IC) constraints correlations $p(b|x, y)$ obtained by exchanging a limited amount of classical information $f(x, a)$, an assumption that cannot be certified in a DI way:



However, a **sufficient condition** for IC violation can be obtained in a DI way by performing a *purely space-like* test, i.e. a Bell test:



After $p(a, b|x, y)$ is collected, one checks if there exists a postprocessing $y, b, f(x, a) \to b'$ such that $p(b'|x, y)$ violates IC.

No-hypersignaling principle

Motivation: constraining space-like correlations or space- and time-like correlations does not single out quantum theory: we push these ideas further by constraining *purely time-like* correlations.

Our general **program**: to derive a theoretical framework to characterize the time-like correlations compatible with any GPT.

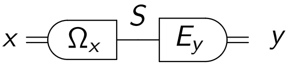
Applictions: we fully characterize all the bipartite extension of a sqit. We show that one such extension is such that:

- does not contradict classical and quantum theories at the level of **space-like** correlations,
- displays an anomalous behavior in its **time-like** correlations.

We formalize this anomaly in terms of a novel principle, that we call **no-hypersignaling**, that inherently constrains **time-like** correlations, hence its violations cannot be detected in a DI way.

Signaling dimension

We consider the **purely time-like** setup of a memory in which, upon input of x, Alice prepares system S into state Ω_x and transmits it to herself in the future:



Let $\mathcal{P}_S^{m \to n}$ denote the convex hull generated by all m-input/n-output conditional probability distributions $p_{y|x}$ that can be obtained by transmitting one elementary system S.

We denote with C_d and Q_d a d-dimensional classical system and a quantum system with d-dimensional Hilbert space, respectively.

Frenkel and Weiner's theorem

Frenkel and Weiner recently proved this remarkable result[1]:

$$\mathcal{P}_{C_d}^{m\to n} = \mathcal{P}_{Q_d}^{m\to n}, \quad \forall m, n.$$

Comparison with the **Holevo bound**: the Holevo bound states that the mutual information attainable by exchanging a d-dimensional classical system and a quantum system with Hilbert-space dimension d are the same.

The Holevo bound is a statement about a *specific function* (mutual information), while Frenkel-Weiner's theorem is about the set of attainable correlations.

Hence, the Holevo bound follows from Frenkel-Weiner's theorem as an immediate **corollary**!

[1] P.E. Frenkel and M. Weiner, Commun. Math. Phys. **340**, 563 (2015)

This motivates us to provide the following **operational** definition of dimension:

Definition (Signaling dimension)

The signaling dimension of a system S, denoted by $\kappa(S)$, is defined as the smallest integer d such that $\mathcal{P}_S^{m\to n} \subseteq \mathcal{P}_{C_d}^{m\to n}$, $\forall m, n$.

Properties of the signaling dimension:

- by definition, $\kappa(S)$ equals the usual classical dimension,
- $\kappa(S)$ also equals [2] the usual quantum dimension, thus for brevity $\mathcal{P}_d^{m\to n} := \mathcal{P}_{C_d}^{m\to n} = \mathcal{P}_{Q_d}^{m\to n}$,
- $\kappa(S)$ does not depend on an arbitrarily made choice of a specific protocol (such as perfect state discrimination);
- $\kappa(S)$ is non-trivial even for those theories where perfectly discriminable states do not exist, i.e. when the no-restriction hypothesis is relaxed.

[2] P.E. Frenkel and M. Weiner, Commun. Math. Phys. **340**, 563 (2015)

No-hypersignaling principle

Analogously, we introduce the following for the signaling dimension:

Definition (No-hypersignaling principle)

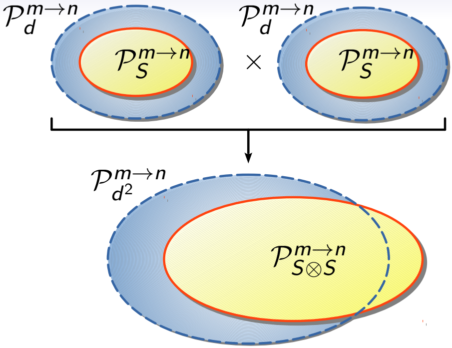
A theory is non-hypersignaling iff, for any set of systems $\{S_k\}$ *with signaling dimensions* $\kappa(S_k)$, *the signaling dimension of the composite system* $\otimes_k S_k$ *satisfies*

$$\kappa(\otimes_k S_k) \leq \prod_k \kappa(S_k).$$

Informally, in terms of input-output correlations, it must not matter if the systems are transmitted separately or jointly.

For two identical systems, no-hypersignaling becomes:

$$\mathcal{P}_S^{m \to n} \subseteq \mathcal{P}_d^{m \to n} \implies \mathcal{P}_{S^{\otimes 2}}^{m \to n} \subseteq \mathcal{P}_{d^2}^{m \to n},$$



An example of a **hypersignaling theory**: while system S satisfies $\mathcal{P}_S^{m\to n} \subseteq \mathcal{P}_d^{m\to n}$, and thus has signaling dimension d, the composite system $S \otimes S$ has a signaling dimension strictly larger than d^2.

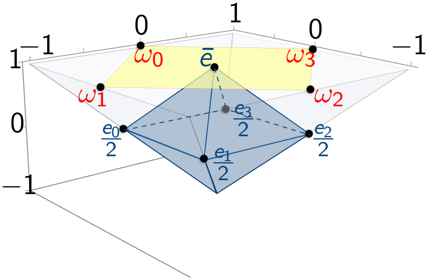
Informally, by transmitting the systems jointly rather than separately, better correlations are achieved.

Hence, by the hyperplane separation theorem, any violation of the no-hypersignaling principle can be detected by a **game** g:

$$g^T \cdot p > \max_{q \in \mathcal{P}_K^{m\to n}} g^T \cdot q, \qquad K = \prod_k \kappa(S_k).$$

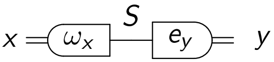
A toy model theory

We start from the **sqit**, the elementary system of the theory commonly considered to produce PR correlations:



- four extremal **states** $\{\omega_x\}$ (yellow square),
- four extremal **effects** $\{e_y\}$, plus the null and unit effects $0, \bar{e}$ (blue cone).

For a sqit S one has that $\mathcal{P}_S^{m \to n} = \mathcal{P}_2^{m \to n}$, that is any correlation $p_{y|x}$ achievable by transferring a sqit:



is also achievable by transferring a classical bit.

All bipartite extensions of a sqit can be given in terms of:

- 24 extremal bipartite states, 8 of which are entangled,
- 24 extremal bipartite effects, 8 of which are entangled.

We derived all the self-consistent bipartite extensions of a sqit:

1. **PR model:** All the 24 states; only the 16 factorized effects;
2. **HS model:** Only the 16 factorized states; all the 24 effects;
3. **Hybrid models:** Two entangled states and effects included;
4. **Frozen Models:** Only one entangled state and effect included, but no allowed reversible dynamics.

Since, for PR model, extremal measurements have four effects, $\mathcal{P}_{S\otimes S}^{m\to n} = \mathcal{P}_{4}^{m\to n}$, i.e. PR model cannot violate no-hypersignaling.

Analogously, since HS model has no entangled states, it cannot exhibit superclassical space-like correlations.

Does the HS model violate the no-hypersignaling principle?

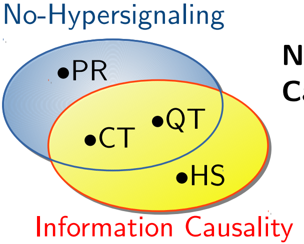
Consider payoff g and input/output correation p, achievable by transmitting a family of seven states $\{\Omega_x\}$ and performing a measurement with seven effects $\{E_y\}$ of HS model:

$$g = \frac{1}{21}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \end{pmatrix}, \quad p = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

By exchanging two classical bits, the optimal payoff is $10/21$, but $g \cdot p = 1/2 > 10/21$, thus no-hypersignaling is **violated**!

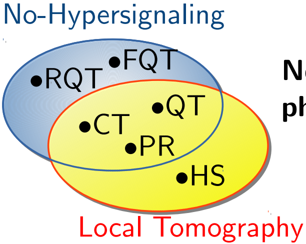
Conclusion

Is no-hypersignaling independent of other principles?



No-Hypersignaling vs Information Causality:

- **CT** classical theory,
- **QT** quantum theory,
- **PR** PR-boxes theory,
- **HS** hypersignaling theory.



No-Hypersignaling vs Local Tomography:

- **RQT** real quantum theory,
- **FQT** fermionic quantum theory.

Motivation: characterizing time-like correlations allowed by QT.

Our general **program**[3]:

- We introduced signaling dimension as an *operational*, *task-independent* dimension for *any* GPT.
- We introduced the no-hypersignaling principle as a *scaling rule* for signaling dimension under system composition.
- We derived a general *theoretical framework* to detect no-hypersignaling violations.

Applications:

- We *fully* characterized *all* the bipartite extensions of a squit.
- By applying our framework, we showed that the HS model's time-like correlations violate the no-hypersignaling principle, but its space-like correlations are compatible with CT and QT.

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[3] M. Dall'Arno, S. Brandsen, A. Tosini, F. Buscemi, and V. Vedral, *No-hypersignaling principle*, Phys. Rev. Lett. **119**, 020401 (2017) (arXiv:1609.09237).