Universal extensions of restricted classes of quantum operations

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*presented by* Christian Gogolin

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If \( \alpha \) is **not a rational multiple** of \( \pi \), by sequences of rotations \( R_\alpha^X, R_\alpha^Y, R_\alpha^Z \) one can approximate arbitrarily well any rotation \( R \) in 3D;

Rotations \( R_\alpha^X, R_\alpha^Y, R_\alpha^Z \) give **full control** over the orientation of an object in \( \mathbb{R}^3 \), or more formally: \( \langle R_\alpha^X, R_\alpha^Y, R_\alpha^Z \rangle = SO(3) \).

More interestingly, \( \langle R_\alpha^X, R_\alpha^Y \rangle = SO(3) \)
Transformations allowed to perform on a quantum system belong the **unitary group** $U(H)$, where $H$ - Hilbert space of the system.

- **Full controllability**: ability to perform any $U \in U(H)$.
- **Limited resources**: only a subset $G \subset U(H)$ is available.
Basic problem (I)

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- **Full controllability**: ability to perform any $U \in U(\mathcal{H})$.
- **Limited resources**: only a subset $G \subset U(\mathcal{H})$ is available.
In this work:

- What gates can be approximated arbitrarily well (generated), when $G$ is supplemented with an additional gate $V \notin G$?
- Physical scenarios considered: restricted gate sets for bosonic and fermionic systems
Examples: distinguishable particles

- **Clifford gates** (important for quantum error-correction) are universal\(^1\) in \((\mathbb{C}^2)^\otimes N\), when supplemented with any extra gate.

- **Local qubit gates** \(LU = U(2) \times U(2) \times \ldots \times U(2)\) plus any entangling gate is universal\(^2\) in \((\mathbb{C}^2)^\otimes N\).

**THIS WORK:** Analogous analysis for indistinguishable particles

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(A) \( \mathcal{N} \) Bosons in \( d \) modes + passive linear optics

\[ \mathcal{H}_b = \text{Sym}^N(\mathbb{C}^d), \quad \text{LO}_b = \{ \mathbf{U}^\otimes N \mid \mathbf{U} \in \text{U}(d) \} \, . \]

(B) \( \mathcal{N} \) Fermions in \( d \) modes + passive linear optics

\[ \mathcal{H}_f = \bigwedge(\mathbb{C}^d), \quad \text{LO}_f = \{ \mathbf{U}^\otimes N \mid \mathbf{U} \in \text{U}(d) \} \, . \]

(C) Fermions in \( d \) modes in the positive parity subspace + active fermionic linear optics

\[ \mathcal{H}^+_{\text{Fock}} = \bigoplus_{m=0}^{\lfloor d/2 \rfloor} \bigwedge^{2m}(\mathbb{C}^d), \quad \text{FLO} - \text{pp. Bogoliubov transformations} \, . \]

Common feature: irreducible representations of compact simple Lie groups
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\textbf{Common feature}: irreducible representations of compact simple Lie groups
What gates can be generated when bosonic/fermionic linear optics is supplemented with an additional gate $V \notin \text{LO}$?

The answer must depend on: type of particles, number of modes and number of particles.
(A) **Bosons**: photonic linear optics\(^3\), interferometry, boson sampling\(^4\) and metrology\(^5\).

(B) **Passive fermionic linear optics**: fermionic interferometry\(^6\), restricted model of quantum computation\(^7\).

(C) **Active fermionic linear optics**: restricted model of quantum computation\(^8\), Ising anyons\(^8\), Matchgates\(^9\).

\(^4\)S. Aaronson and A. Arkhipov, ACM, (2011)
Physical context and motivation

(A) **Bosons**: photonic linear optics\(^3\), interferometry, boson sampling\(^4\) and metrology\(^5\).

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THEOREM

Let $V \notin \text{LO}_b$ be a gate acting on Hilbert space of $N$ bosons in $d$ modes. We have the following possibilities:

(i) If $d > 2$, then $V$-universal, $\langle \text{LO}_b, V \rangle = U(\mathcal{H}_b)$.

(ii) If $d = 2$ and $[V \otimes V, \mathbb{I}_b] \neq 0$, then $V$-universal, $\langle \text{LO}_b, V \rangle = U(\mathcal{H}_b)$.

(iii) If $d = 2$, $N \neq 6$ and $[V \otimes V, \mathbb{I}_b] = 0$, then

$$\langle \text{LO}_b, V \rangle = H_b = \{ U \in U(\mathcal{H}_b) | [U \otimes U, \mathbb{I}_b] = 0 \}.$$ 

(iv) If $d = 2$, $N = 6$, and $[V \otimes V, \mathbb{I}_b] = 0$ - even crazier things happen :)

For $d = 2$ we define $\mathbb{I}_b = |\Psi_b\rangle\langle\Psi_b|$, where

$$|\Psi_b\rangle = \sum_{k=0}^{N} (-1)^k |D_k\rangle |D_{N-k}\rangle \in \mathcal{H}_b \otimes \mathcal{H}_b,$$

and $|D_k\rangle$ are two-mode Dicke states.
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Extensions of $\text{LO}_b$ for $d = 2$ modes and $N \neq 6$ particles.

What is the group $H_b$?

- When $N$ - even, then $H_b = \langle \text{SO}(H_b), \exp(i\phi)\mathbb{I} \rangle$ and we have no transitivity for pure states;

- When $N$ - odd, then $H_b = \langle \text{USp}(H_b), \exp(i\phi)\mathbb{I} \rangle$ and we have transitivity for pure states;

- **Example** Hamiltonian $H_{in} = n_1^3 - n_2^3$ promotes $\text{LO}_b$ to $H_b$. 
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Passive bosonic linear optics (II)

Extensions of $\text{LO}_b$ for $d = 2$ modes and $N \neq 6$ particles.

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Results for fermions

Possible extensions of $L O_f$ for the case of half-filling $2N = d$. 

![Diagram showing possible extensions of $L O_f$](image-url)
Possible extensions of $\text{LO}_f$ for the case of half-filling $2N = d$.

Possible extensions of FLO for even number of physical modes $d = 2n$. 
Random bosonic circuits for quantum metrology

Universal extensions of passive bosonic linear optics can be used to generate random bosonic circuits which generate states useful in quantum metrology\textsuperscript{10}.

\textbf{SQL:} $\Delta \varphi \approx \frac{1}{N}$ \hspace{1cm} \textbf{HL:} $\Delta \varphi \approx \frac{1}{N^2}$

Can states mimicking the properties of Haar-random states on $\mathcal{H}_b$ be generated efficiently?

**Known result:** [F. Brandao, A. Harrow and M. Horodecki 2016] Sufficiently long random circuits formed from the set of gates universal in $\mathcal{H}$ give approximate $t$-designs.

Our strategy: supplement gates universal for linear optics to get universality on $\mathcal{H}_b$ for $d = 2$ modes.

**Construction of the universal set of gates in $\mathcal{H}_b$:**

- **Three linear gates** generating whole linear optics [Sarnak 1986]

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  V_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix},
  V_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},
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- Supplement this set of gates by **cross-Kerr like** transformation

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- Supplement this set of gates by **cross-Kerr like** transformation $V_{CK} = \exp\left(-i \frac{\pi}{3} n_1 n_2 \right)$. 
In each step the gate $U_i$ is chosen uniformly at random from the universal gateset:

$$U_i \in \{ \hat{V}_1, \hat{V}_2, \hat{V}_3, V_{CK}, +h.c \}$$
Numerical results

Standard interferometric setup:

Comparison with analytical results for Haar-random states.
The extension problem analysed and solved for (A) passive bosonic linear optics, (B) passive fermionic linear optics, and (C) active fermionic linear optics.

The possible behaviour is rich and depends on the type of particles, number of modes, and number of particles.

Applications to quantum sensing and FLO model of quantum computation.
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Applications to **quantum sensing** and **FLO model of quantum computation**.
THANK YOU FOR YOUR ATTENTION!

SEE ARXIV:1705.11188 FOR DETAILS
Other classes of restricted gate sets (matchgates, non-parity preserving FLO);

Identification of universal gates that allow for **efficient error correction**;

Which additional gates give the **most efficient** approximation?

Identification of “physical resources” that are responsible for universality (**non-Gaussianity, contextuality**?);

Peculiar case of $N = 6$ for two-mode bosons (**exotic $G_2$ group**...).
A collection of quantum gates $S \subseteq U(\mathcal{H})$ is called \textbf{universal} in $\mathcal{H}$ iff every element $U \in U(\mathcal{H})$ can be \textbf{approximated arbitrarily well} with elements $U_i \in S$:

$$\forall \epsilon \exists U_{i_k} \in S \text{ such that } \| U - U_{i_1}U_{i_2}\cdots U_{i_N} \| \leq \epsilon$$

\textbf{Notation:} $\langle S \rangle$ - the group of transformations (approximately) generated by $S$.

\textbf{Motivation:} quantum computing and control of quantum systems
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Formal definition of universality

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Two mode transformations are universal\textsuperscript{11} in $\mathbb{C}^d$.

Any non-trivial two-mode beam-splitter is universal\textsuperscript{12} in $\mathbb{C}^d$.

Three-mode beam-splitters and beyond\textsuperscript{13}.


\textsuperscript{12}A. Bouland and S. Aaronson, Phys. Rev. A \textbf{89}, 062316 (2014)

\textsuperscript{13}A. Sawicki, Quant. Inf. Comp. \textbf{16}, No. 3-4, 0291-0312 (2016)
Limited gate sets considered are closely related to irreducible representations of **simple compact Lie groups** $SU(d)$ (passive LO) and $Spin(2d)$ (active LO).

$$\Pi : G \to U(\mathcal{H}), \quad \Pi(g_1 g_2) = \Pi(g_1) \Pi(g_2).$$

This allows to use the machinery of Lie algebras$^{14}$: linear spaces of the corresponding **hermitian generators**.

**Main technical tool:** Dynkin classification$^{15}$ of **maximal Lie subalgebras of simple Lie algebras**.

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$^{15}$ Selected papers of E. B. Dynkin with commentary, AMS (2000)
• **Special orthogonal group** $\text{SO}(\mathcal{H})$ consists of unitaries preserving **bilinear non-degenerate symmetric form** $B_s$

\[ B_s(\langle \psi |, \langle \phi |) = B_s(\langle \phi |, \langle \psi |), \quad B_s(U \langle \psi |, U \langle \phi |) = B_s(\langle \psi |, \langle \phi |) \]

• **Symplectic unitary group** $\text{USp}(\mathcal{H})$ consists of unitaries preserving **bilinear non-degenerate antisymmetric form** $B_a$

\[ B_a(\langle \psi |, \langle \phi |) = -B_a(\langle \phi |, \langle \psi |), \quad B_a(U \langle \psi |, U \langle \phi |) = B_a(\langle \psi |, \langle \phi |) \]

• $\text{USp}(\mathcal{H})$ is possible iff $|\mathcal{H}|$-even. Moreover, $\text{USp}(\mathcal{H})$ acts transitively on pure states in $\mathcal{H}$.
Passive fermionic linear optics

For $d = 2N$ (half-filling) we define $\mathbb{L}_f = |\Psi_f\rangle\langle \Psi_f|$, where

$$|\Psi_f\rangle = |1\rangle \wedge |2\rangle \wedge \ldots \wedge |2N\rangle \in \mathcal{H}_f \otimes \mathcal{H}_f.$$
Extensions of $\text{LO}_f$ for the case of half-filling $2N = d$.

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- When $N$ - odd, then $H_f = \langle \text{USp}(\mathcal{H}_f), \exp(i\phi)\mathbb{I} \rangle$ and we have **transitivity for pure states**;

**Example**: Correlated hopping Hamiltonian\textsuperscript{16}

\[ H_{in}' = a_1^\dagger (\hat{n}_2 - \hat{n}_3)^2 a_4 + \text{h.c.} \]

promotes $\text{LO}_f$ to $H_f$.

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\textsuperscript{16} F. Dolcini and A. Montorsi, Phys. Rev. B 88, 115115 (2013)
For $d = 2n$ we define:

$$\mathbb{L}_{\text{FLO}} = \frac{1}{2^{d(2d-1)}} \prod_{1 \leq i < j \leq 2d} \left( \mathbb{I} \otimes \mathbb{I} + m_i m_j \otimes m_i m_j \right).$$

**Theorem**

Let $V \notin \text{FLO}$ be a gate acting on positive-parity subspace of Fermionic $d$-mode Fock space $\mathcal{H}^{+}_{\text{Fock}}$. We have the following possibilities:

(i) If $d \neq 2n$, then $\langle \text{FLO}, V \rangle = U \left( \mathcal{H}^{+}_{\text{Fock}} \right)$

(ii) If $d = 2n$, and $[V \otimes V, \mathbb{L}_{\text{FLO}}] = 0$, then

$$\langle \text{FLO}, V \rangle = H_{\text{FLO}} = \{ U \in U \left( \mathcal{H}^{+}_{\text{Fock}} \right) | [U \otimes U, \mathbb{L}_{\text{FLO}}] = 0 \}.$$

(iii) If $d = 2n$ and $[V \otimes V, \mathbb{L}_{\text{FLO}}] \neq 0$, then $\langle \text{FLO}, V \rangle = U \left( \mathcal{H}^{+}_{\text{Fock}} \right)$. 
Extensions of FLO for even number of physical modes $d = 2n$. 

- When $n$ - even, then $H_{\text{FLO}} = \langle \text{SO}(\mathcal{H}_{\text{Fock}}^+), \exp(i\phi)\mathbb{I} \rangle$ and we have no transitivity for pure states;

- When $n$ - odd, then $H_{\text{FLO}} = \langle \text{USp}(\mathcal{H}_{\text{Fock}}^+), \exp(i\phi)\mathbb{I} \rangle$ and we have transitivity for pure states;
Extensions of FLO for even number of physical modes $d = 2n$.

- When $n$ - even, then $H_{\text{FLO}} = \langle \text{SO}(H_{\text{Fock}}^+), \exp(i\phi)\mathbb{I}\rangle$ and we have no transitivity for pure states;
- When $n$ - odd, then $H_{\text{FLO}} = \langle \text{USp}(H_{\text{Fock}}^+), \exp(i\phi)\mathbb{I}\rangle$ and we have transitivity for pure states;