Universal extensions of restricted classes of quantum operations

Michał Oszmaniec and Zoltán Zimborás
presented by Christian Gogolin

AQIS'17, 8 September 2017

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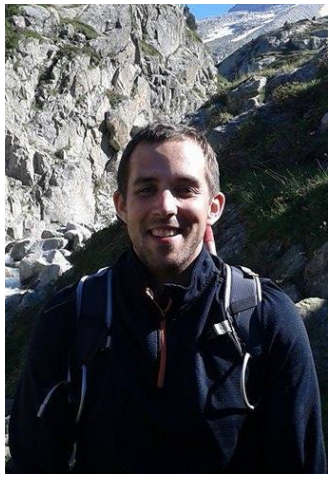

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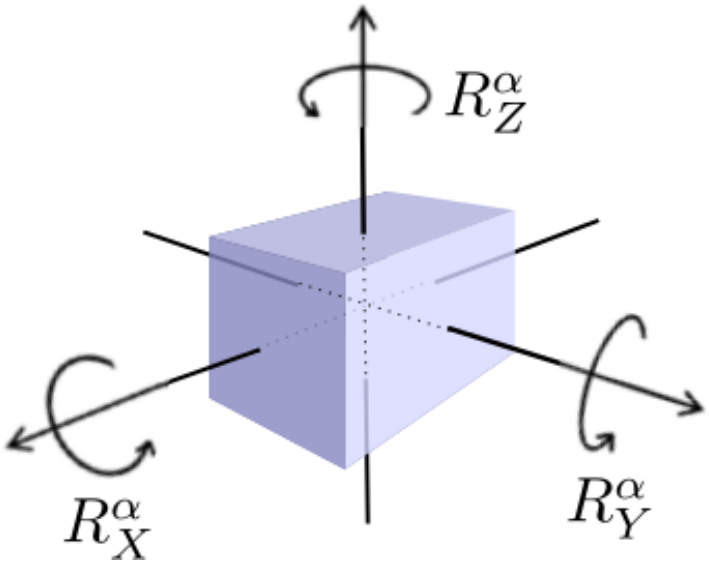
Simple example: Rotations in 3D



- If α is **not a rational multiple** of π, by sequences of rotations $R_X^\alpha, R_Y^\alpha, R_Z^\alpha$ one can approximate arbitrarily well **any** rotation R in 3D;
- Rotations $R_X^\alpha, R_Y^\alpha, R_Z^\alpha$ give **full control** over the orientation of an object in $\mathbb{R}^3$, or more formally: $\langle R_X^\alpha, R_Y^\alpha, R_Z^\alpha \rangle = \mathrm{SO}(3)$.
- More interestingly, $\langle R_X^\alpha, R_Y^\alpha \rangle = \mathrm{SO}(3)$

Basic problem (I)

- Transformations allowed to perform on a quantum system belong **the unitary group** $\mathrm{U}(\mathcal{H})$, where $\mathcal{H}$ - Hilbert space of the system.
- **Full controllability:** ability to perform any $U \in \mathrm{U}(\mathcal{H})$.
- **Limited resources**: only a subset $G \subset \mathrm{U}(\mathcal{H})$ is available.

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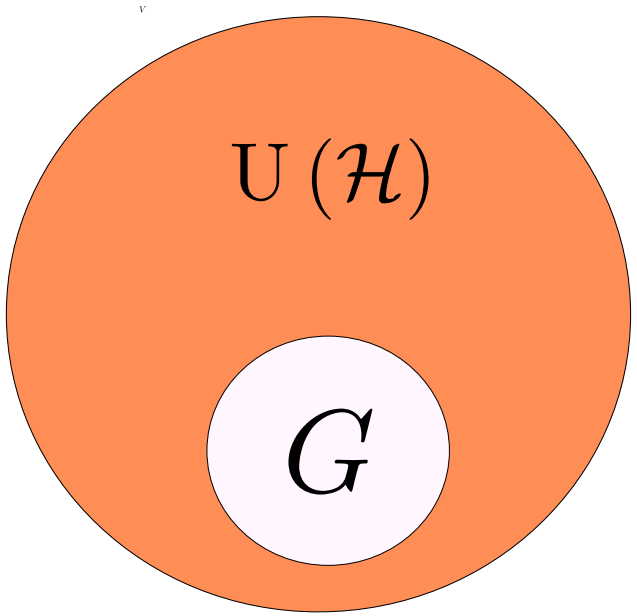
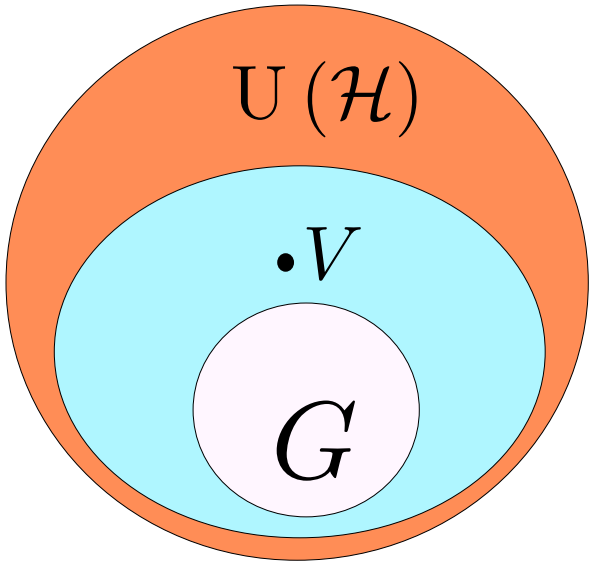


Basic problem (II)

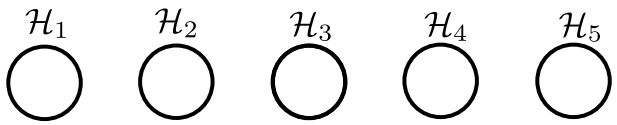
In **this work**:

- What gates can be approximated arbitrarily well (generated), when G is **supplemented with and additional gate** $V \notin G$?
- Physical scenarios considered: restricted gate sets for **bosonic and fermionic** systems



$$\langle G, V \rangle = ?$$

Examples: distinguishable particles



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- **Local qubit gates** $\mathrm{LU} = \mathrm{U}(2) \times \mathrm{U}(2) \times \ldots \times \mathrm{U}(2)$ plus any entangling gate is universal[2] in $(\mathbb{C}^2)^{\otimes N}$.

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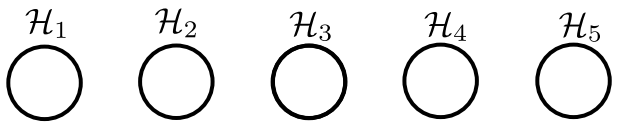

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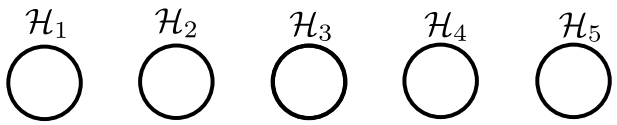

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SETTING

(A) N **Bosons** in d modes + *passive* linear optics

$$\mathcal{H}_b = \mathrm{Sym}^N(\mathbb{C}^d)\ ,\ \mathrm{LO}_b = \left\{ U^{\otimes N} \mid U \in \mathrm{U}(d) \right\}\ .$$

(B) N **Fermions** in d modes + *passive* linear optics

$$\mathcal{H}_f = \bigwedge^N(\mathbb{C}^d)\ ,\ \mathrm{LO}_f = \left\{ U^{\otimes N} \mid U \in \mathrm{U}(d) \right\}\ .$$

(C) **Fermions** in d modes in the positive parity subspace + *active* fermionic linear optics

$$\mathcal{H}^+_{\mathrm{Fock}} = \bigoplus_{m=0}^{\lfloor \frac{d}{2} \rfloor} \bigwedge^{2m}(\mathbb{C}^d)\ ,\ \mathrm{FLO}\text{ - pp. Bogoliubov transformations}\ .$$

Common feature: irreducible representations of compact simple Lie groups

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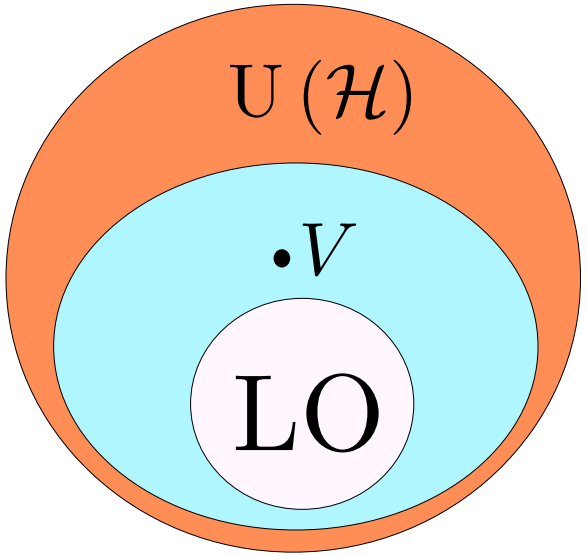
Problem studied

What gates can be generated when bosonic/fermionic linear optics is **supplemented with an additional gate** $V \notin \mathrm{LO}$?



$\langle \mathrm{LO}, V \rangle = ?$

The answer must depend on: **type of particles**, **number of modes** and **number of particles**.

Physical context and motivation

(A) **Bosons**: photonic linear optics[3], interferometry, boson sampling[4] and metrology[5].

(B) **Passive fermionic linear optics**: fermionic interferometry[6], restricted model of quantum computation[7].

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[6] E. Bocquillon, *et al.*, Annalen der Physik **526**, 1 (2014
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Passive bosonic linear optics (I)

THEOREM

Let $V \notin \mathrm{LO}_b$ be a gate acting on Hilbert space of N bosons in d modes. We have the following possibilities:

(i) If $d > 2$, then V-**universal**, $\langle \mathrm{LO}_b, V \rangle = \mathrm{U}(\mathcal{H}_b)$.

(ii) If $d = 2$ and $[V \otimes V, \mathbb{L}_b] \neq 0$, then V-**universal**, $\langle \mathrm{LO}_b, V \rangle = \mathrm{U}(\mathcal{H}_b)$.

(iii) If $d = 2$, $N \neq 6$ and $[V \otimes V, \mathbb{L}_b] = 0$, then

$$\langle \mathrm{LO}_b, V \rangle = H_b = \{ U \in \mathrm{U}(\mathcal{H}_b) | \, [U \otimes U, \mathbb{L}_b] = 0 \}.$$

(iv) If $d = 2$, $N = 6$, and $[V \otimes V, \mathbb{L}_b] = 0$ - even crazier things happen :)

For $d = 2$ we define $\mathbb{L}_b = |\Psi_b\rangle\langle\Psi_b|$, where

$$|\Psi_b\rangle = \sum_{k=0}^{N} (-1)^k |D_k\rangle |D_{N-k}\rangle \in \mathcal{H}_b \otimes \mathcal{H}_b \ ,$$

and $|D_k\rangle$ are two-mode Dicke states.

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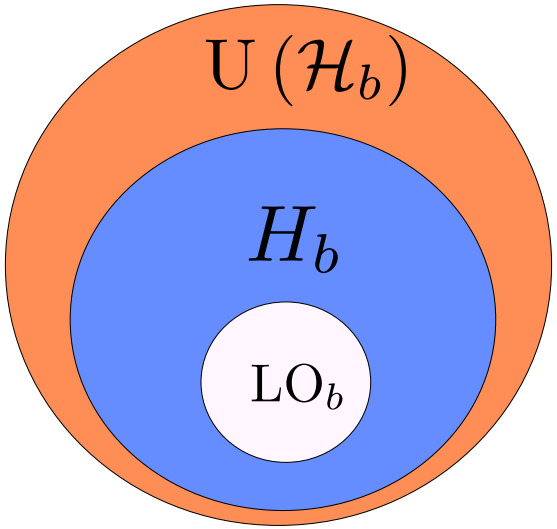

What is the group H_b?

- When N - even, then $H_b = \langle \mathrm{SO}(\mathcal{H}_b), \exp(i\phi)\mathbb{I} \rangle$ and we have **no transitivity for pure states**;
- When N - odd, then $H_b = \langle \mathrm{USp}(\mathcal{H}_b), \exp(i\phi)\mathbb{I} \rangle$ and we have **transitivity for pure states**;
- **Example** Hamiltonian $H_{in} = n_1^3 - n_2^3$ promotes LO_b to H_b.

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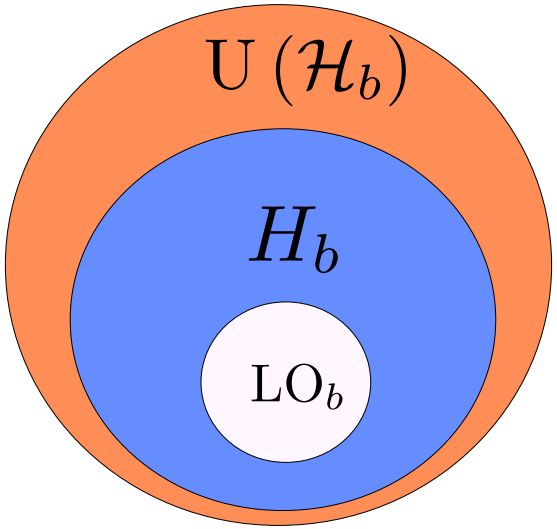

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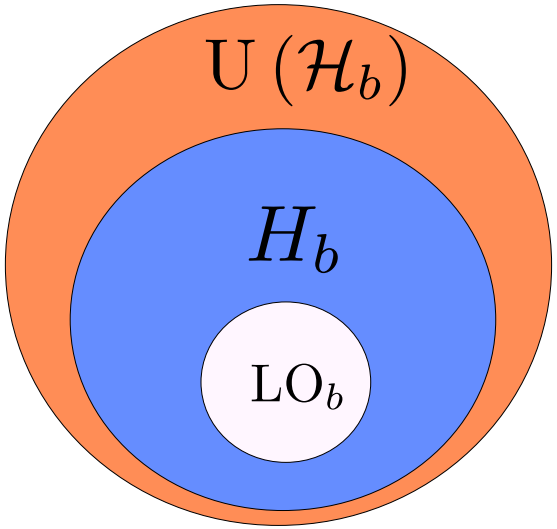

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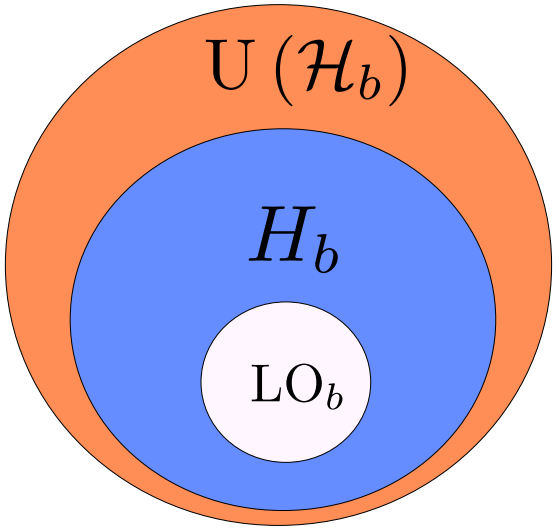

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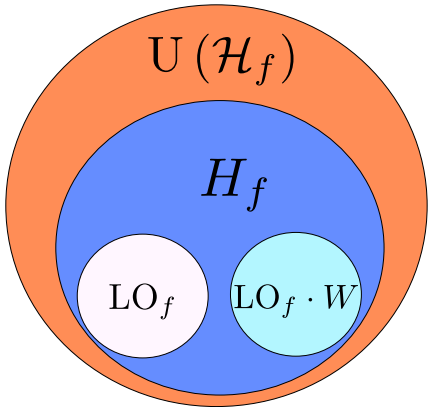
Results for fermions

Possible extensions of LO_f for the case of half-filling $2N = d$.



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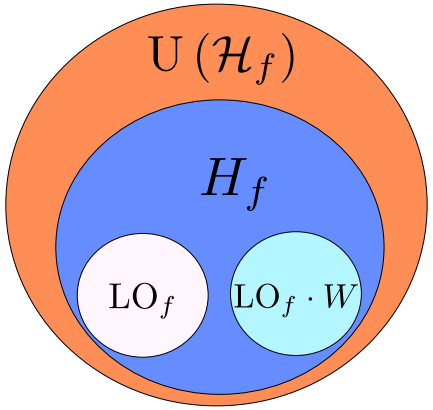
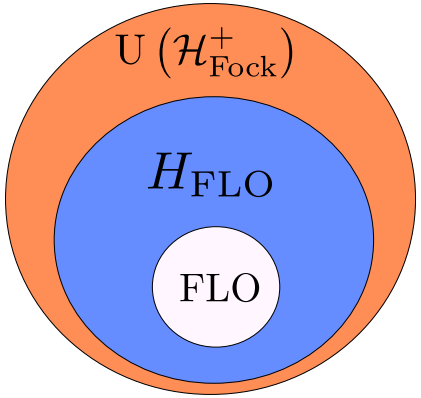
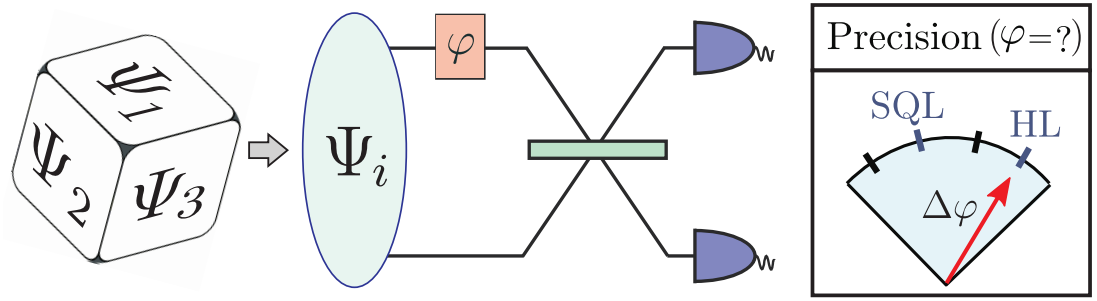


Possible extensions of FLO for even number of physical modes $d = 2n$.



Random bosonic circuits for quantum metrology



SQL: $\Delta\varphi \approx \frac{1}{N}$ HL: $\Delta\varphi \approx \frac{1}{N^2}$

Universal extensions of passive bosonic linear optics can be used to generate random bosonic circuits which generate states useful in quantum metrology[10].

[10] M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acin, and M. Lewenstein Phys. Rev. X **6**, 041044 (2016)

Random bosonic circuits for quantum metrology (II)

- Can states mimicking the properties of Haar-random states on $\mathcal{H}_b$ be **generated efficiently**?
- **Known result**: [F. Brandao, A. Harrow and M. Horodecki 2016] Sufficiently long random circuits formed from the set of gates universal in $\mathcal{H}$ give approximate t-designs.
- Our strategy: supplement gates universal for linear optics to get universality on $\mathcal{H}_b$ for $d = 2$ modes.

Construction of the universal set of gates in $\mathcal{H}_b$:

- **Three linear gates** generating whole linear optics [Sarnak 1986]

$$V_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, V_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

$$V_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix},$$

- Supplement this set of gates by **cross-Kerr like** transformation $V_{\mathrm{CK}} = \exp\left(-i\frac{\pi}{3} n_1 n_2\right)$.

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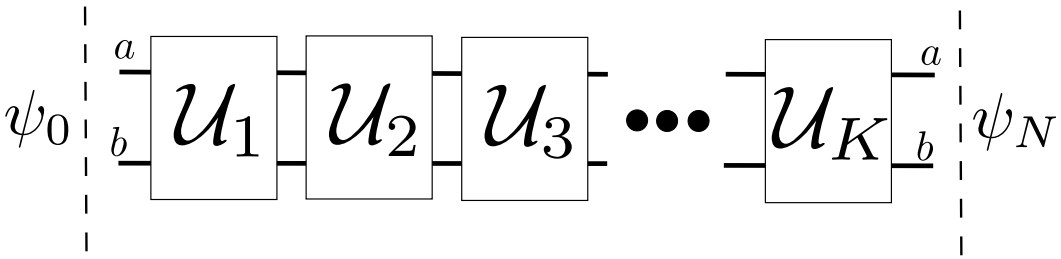
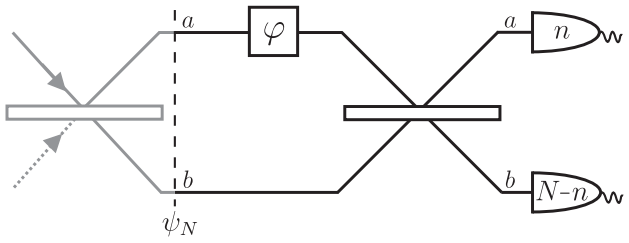


In each step the gate $\mathcal{U}_i$ is chosen uniformly at random from the universal gateset:

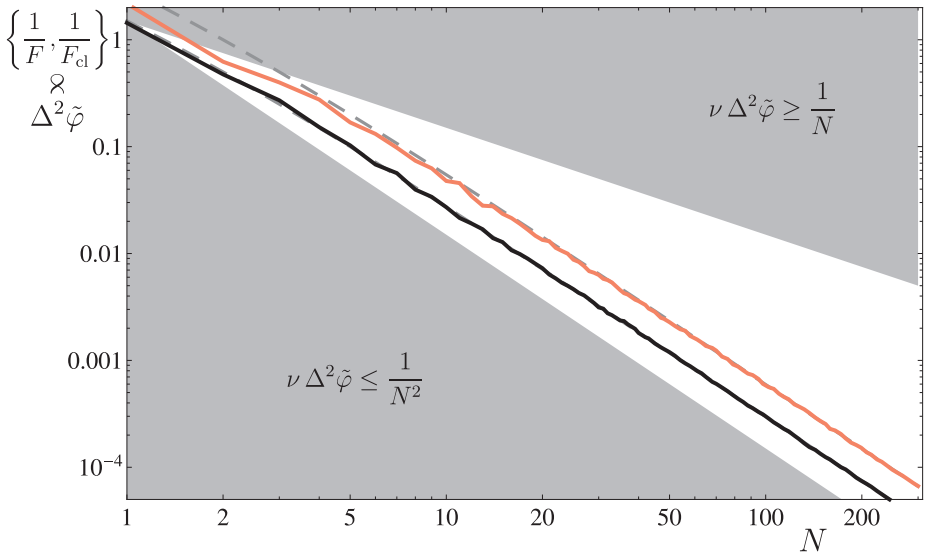
$$\mathcal{U}_i \in \left\{\hat{V}_1, \hat{V}_2, \hat{V}_3, V_{\mathsf{CK}}, +\mathsf{h.c}\right\}$$

Numerical results

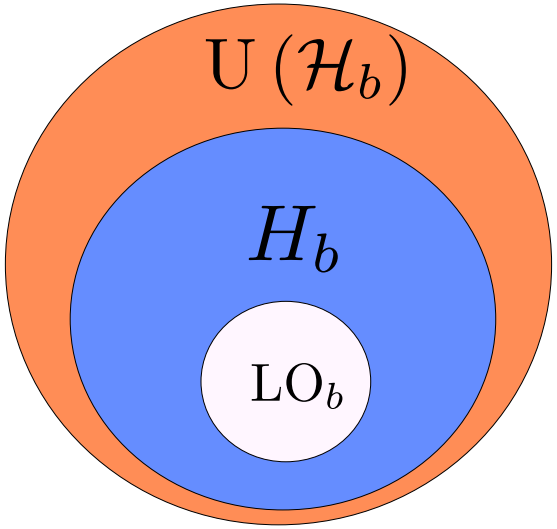
Standard interferometric setup:



Comparison with analytical results for Haar-random states.



SUMMARY



- The **extension problem analysed** and solved for (A) passive bosonic linear optics, (B) passive fermionic linear optics, and (C) active fermionic linear optics.
- The possible behaviour is **rich** and depends on the type of particles, number of modes, and number of particles.
- Applications to **quantum sensing** and **FLO model of quantum computation**.

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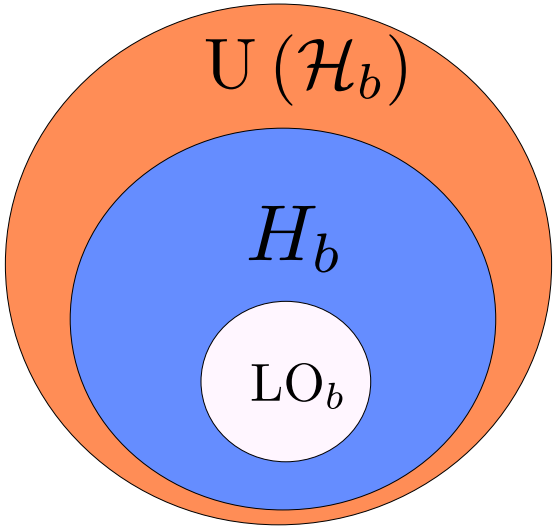

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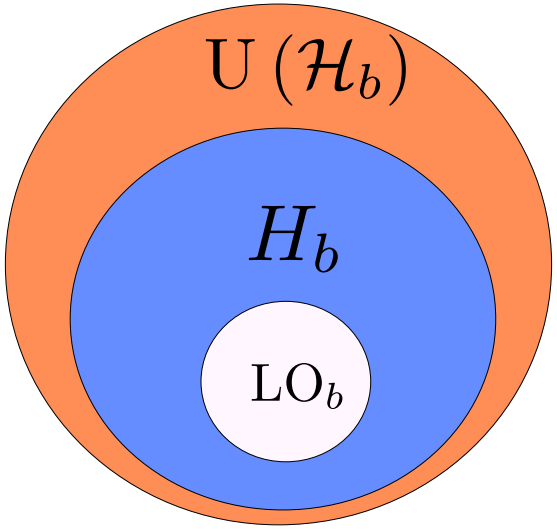

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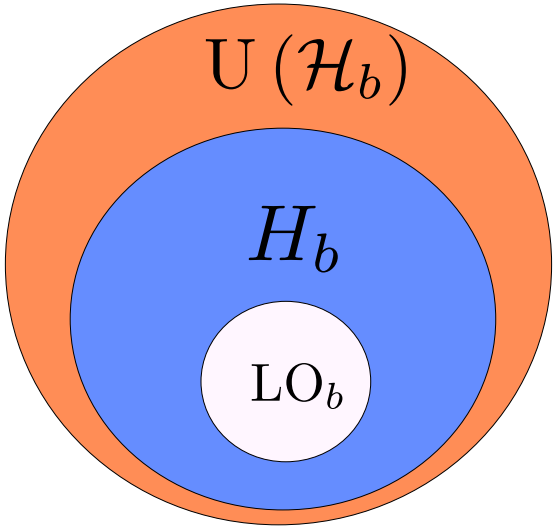

THANK YOU FOR YOUR ATTENTION!

SEE `ARXIV:1705.11188` **FOR DETAILS**

OPEN PROBLEMS

- Other classes of restricted gate sets (matchgates, non-parity preserving FLO);
- Identification of universal gates that allow for **efficient error correction**;
- Which additional gates give the **most efficient** approximation?
- Identification of "physical resources" that are responsible for universality (**non-Gaussianity**, **contextuality**?);
- Peculiar case of $N = 6$ for two-mode bosons (**exotic** G_2 **group**...)

Formal definition of universality

- A collection of quantum gates $\mathcal{S} \subset \mathrm{U}(\mathcal{H})$ is called **universal** in $\mathcal{H}$ iff every element $U \in \mathrm{U}(\mathcal{H})$ can be **approximated arbitrarily well** with elements $U_i \in \mathcal{S}$:

$$\forall_\epsilon \ \exists_{U_{i_k} \in \mathcal{S}} \text{ such that } \|U - U_{i_1} U_{i_2} \cdots U_{i_N}\| \leq \epsilon$$

- **Notation**: $\langle \mathcal{S} \rangle$ - the group of transformations (approximately) generated by $\mathcal{S}$.
- **Motivation**: quantum computing and control of quantum systems

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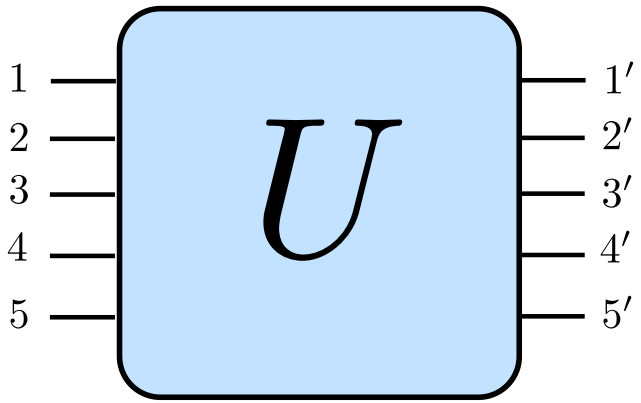
Example: linear optics



- Two mode transformations are universal[11] in $\mathbb{C}^d$.
- Any non-trivial two-mode beam-splitter is universal[12] in $\mathbb{C}^d$.
- Three-mode beam-splitters and beyond[13].

[11] M. Roek *et al.*, Phys. Rev. Lett. **73**, 58 (1994)
[12] A. Bouland and S. Aaronson, Phys. Rev. A **89**, 062316 (2014)
[13] A. Sawicki, Quant. Inf. Comp. **16**, No. 3-4, 0291-0312 (2016)

Mathematical detour (I)

- Limited gate sets considered are closely related to irreducible representations of **simple compact Lie groups** $\mathrm{SU}(d)$ (passive LO) and $\mathrm{Spin}(2d)$ (active LO).

$$\Pi : G \to \mathrm{U}(\mathcal{H}) \ , \ \Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2) \ .$$

- This allows to use the machinery of Lie algebras[14]: linear spaces of the corresponding **hermitian generators**.
- Main technical tool: Dynkin classification[15] of **maximal Lie subalgebras of simple Lie algebras**.

[14] Zoltan Zimboras *et al.*, Phys. Rev. A **92**, 042309 (2015)
[15] Selected papers of E. B. Dynkin with commentary, AMS (2000)

Mathematical detour (II)

- **Special orthogonal group** $\mathrm{SO}(\mathcal{H})$ consists of unitaries preserving **bilinear non-degenerate symmetric form** B_s

$$B_s(|\psi\rangle, |\phi\rangle) = B_s(|\phi\rangle, |\psi\rangle) \ , \ B_s(U|\psi\rangle, U|\phi\rangle) = B_s(|\psi\rangle, |\phi\rangle)$$

- **Symplectic unitary group** $\mathrm{USp}(\mathcal{H})$ consists of unitaries preserving **bilinear non-degenerate antisymmetric form** B_a

$$B_a(|\psi\rangle, |\phi\rangle) = -B_a(|\phi\rangle, |\psi\rangle) \ , \ B_a(U|\psi\rangle, U|\phi\rangle) = B_a(|\psi\rangle, |\phi\rangle)$$

- $\mathrm{USp}(\mathcal{H})$ is possible iff $|\mathcal{H}|$-**even**. Moreover, $\mathrm{USp}(\mathcal{H})$ **acts transitively on pure states in** $\mathcal{H}$.

Passive fermionic linear optics

For $d = 2N$ (half-filling) we define $\mathbb{L}_f = |\Psi_f\rangle\langle\Psi_f|$, where

$$|\Psi_f\rangle = |1\rangle \wedge |2\rangle \wedge \ldots \wedge |2N\rangle \in \mathcal{H}_f \otimes \mathcal{H}_f.$$

Theorem

Let $V \notin \mathrm{LO}_f$ be a gate acting on Hilbert space of N fermions in d modes. We have the following possibilities:

(i) If $d \neq 2N$, then $\langle \mathrm{LO}_f, V\rangle = \mathrm{U}(\mathcal{H}_f)$.

(ii) If $d = 2N$ and $V = Wg$, for $g \in \mathrm{LO}_f$ and $W = \prod_{i=1}^{d}(a_i + a_i^\dagger)$, then

$$\langle \mathrm{LO}_f, V\rangle = \mathrm{LO}_f \cup \mathrm{LO}_f \cdot W \ .$$

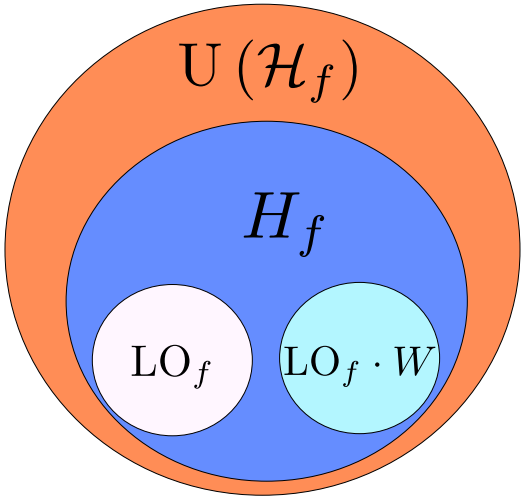
(iii) If $d = 2N$, $V \neq gW$, for $g \in \mathrm{LO}_f$, and $[V \otimes V, \mathbb{L}_f] = 0$, then

$$\langle \mathrm{LO}_f, V\rangle = H_f = \{U \in \mathrm{U}(\mathcal{H}_f) |\ [V \otimes V, \mathbb{L}_f] = 0\ \}.$$

(iv) If $d = 2N$ and $[V \otimes V, \mathbb{L}_f] \neq 0$, then $\langle \mathrm{LO}_f, V\rangle = \mathrm{U}(\mathcal{H}_f)$.

Passive fermionic linear optics

Extensions of LO_f for the case of half-filling $2N = d$.



- When N - even, then $H_f = \langle \mathrm{SO}(\mathcal{H}_f), \exp(i\phi)\mathbb{I} \rangle$ and we have **no transitivity for pure states**;
- When N - odd, then $H_f = \langle \mathrm{USp}(\mathcal{H}_f), \exp(i\phi)\mathbb{I} \rangle$ and we have **transitivity for pure states**;
- **Example**: Correlated hopping Hamiltonian[16] $H'_{in} = a_1^\dagger (\hat{n}_2 - \hat{n}_3)^2 a_4 + \mathrm{h.c}$ promotes LO_f to H_f.

[16] F. Dolcini and A. Montorsi, Phys. Rev. B 88, 115115 (2013)

Active fermionic linear optics

For $d = 2n$ we define:

$$\mathbb{L}_{\mathrm{FLO}} = \frac{1}{2^{d(2d-1)}} \prod_{1 \leq i < j \leq 2d} \left(\mathbb{I} \otimes \mathbb{I} + m_i m_j \otimes m_i m_j \right) \ .$$

Theorem

Let $V \notin \mathrm{FLO}$ be a gate acting on positive-parity subspace of Fermionic d-mode Fock space $\mathcal{H}^{+}_{\mathrm{Fock}}$. We have the following possibilities:

(i) If $d \neq 2n$, then $\langle \mathrm{FLO}, V \rangle = \mathrm{U}\left(\mathcal{H}^{+}_{\mathrm{Fock}}\right)$

(ii) If $d = 2n$, and $[V \otimes V, \mathbb{L}_{\mathrm{FLO}}] = 0$, then

$$\langle \mathrm{FLO}, V \rangle = H_{\mathrm{FLO}} = \{ U \in \mathrm{U}\left(\mathcal{H}^{+}_{\mathrm{Fock}}\right) \big| \, [U \otimes U, \mathbb{L}_{\mathrm{FLO}}] = 0 \ \}.$$

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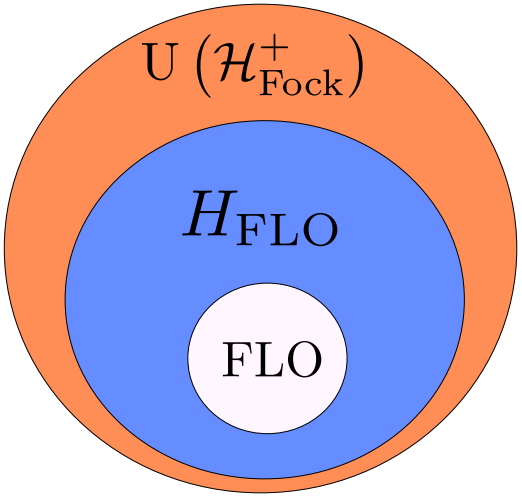
Active fermionic linear optics (II)

Extensions of FLO for even number of physical modes $d = 2n$.



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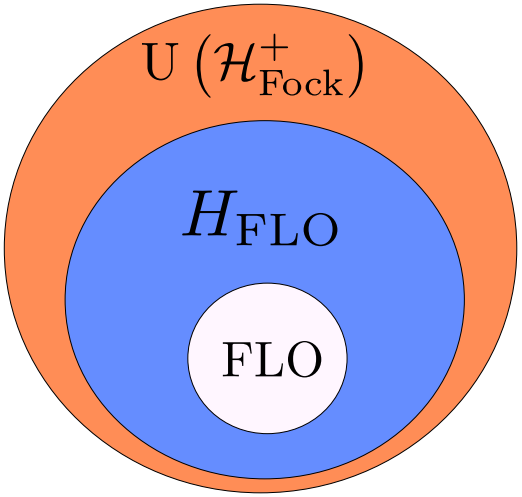

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