Constructing optimal quantum error correcting codes from absolutely maximally entangled states

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arXiv:1701.03359

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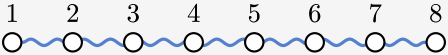
What are AME states?

$|\psi\rangle =$ 

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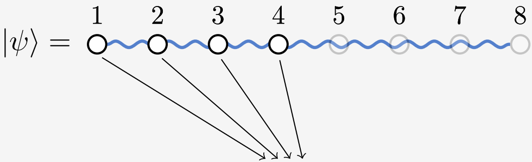

$$\mathrm{Tr}_{\{1,2,3,4\}^c} |\psi\rangle\langle\psi| \propto \mathbb{1}$$

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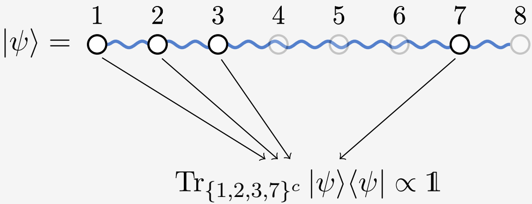

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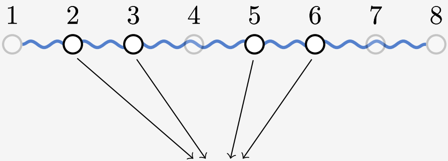


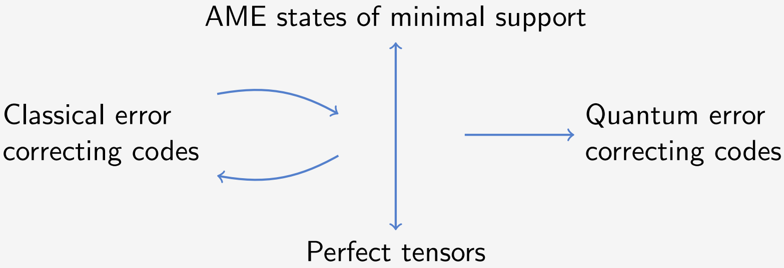
$$\mathrm{Tr}_{\{2,3,5,6\}^c} |\psi\rangle\langle\psi| \propto \mathbb{1}$$

AME states

A state of n particles is AME if for all $S \subset \{1, \ldots, n\}$

$$|S| \leq \lfloor n/2 \rfloor \implies \mathrm{Tr}_{S^c} |\psi\rangle\langle\psi| \propto \mathbb{1}.$$

Content of this talk



[1] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Phys. Rev. A, 92 (3 2015), 032316
[2] W. Helwig and W. Cui, (2013), URL: https://arxiv.org/abs/1306.2536

Why are AME states interesting?

- Natural generalization of EPR and GHZ states

[3] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.-K. Lo, Phys. Rev. A, 86 (5 2012), 052335
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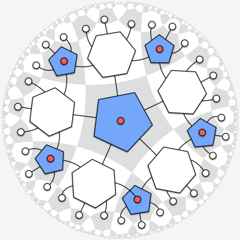

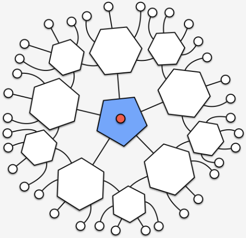

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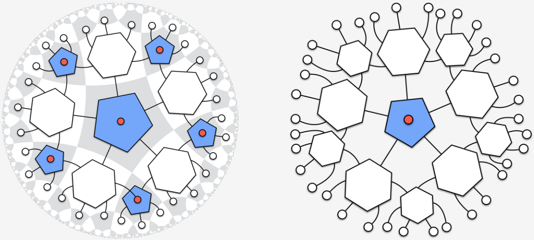

- Still fundamental questions open. Existence (qubits):
  $n = 2, 3, \not{4}, 5, 6, 7, \not{8}, \not{9}, \ldots$ [5, 6]

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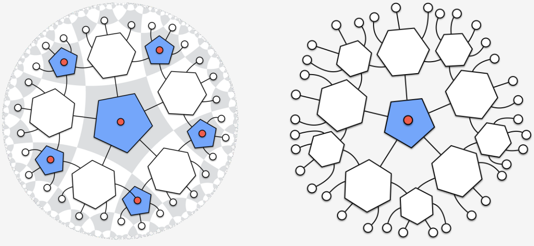

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Classical error correcting codes

Message

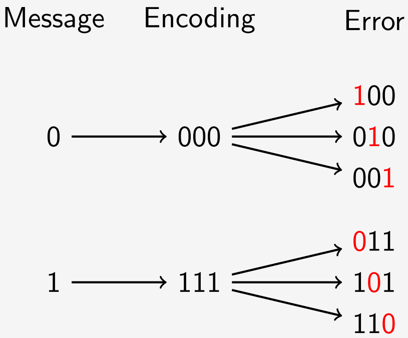
0

1

Classical error correcting codes

Message	Encoding
$0 \longrightarrow$	000
$1 \longrightarrow$	111

Classical error correcting codes



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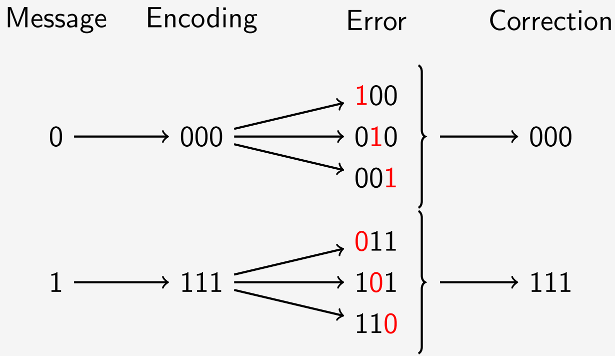

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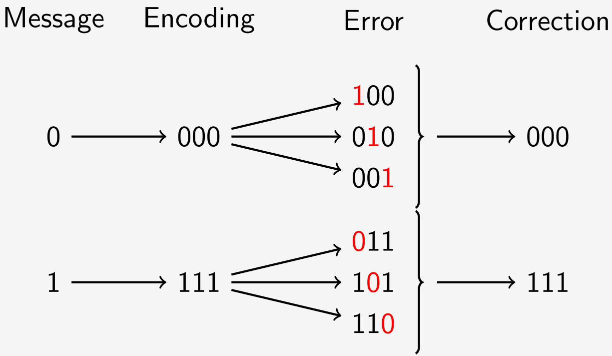

$$[\ n = 3\ ,\ k = 1\ ,\ d_H = 3\ ]_{\ q = 2}$$

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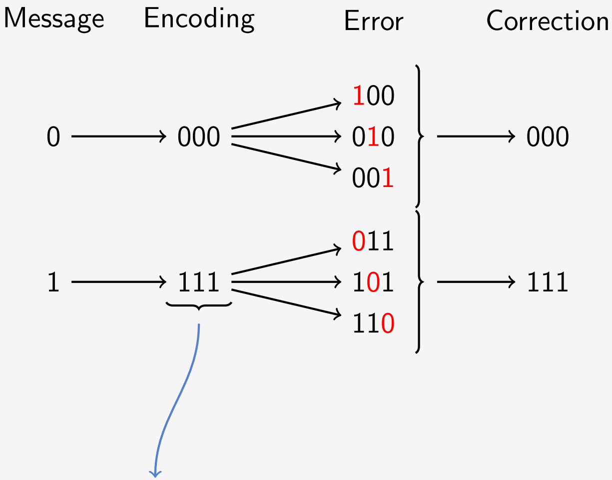

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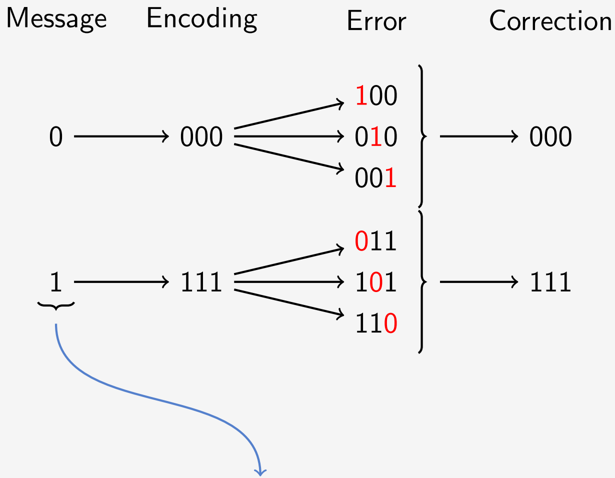

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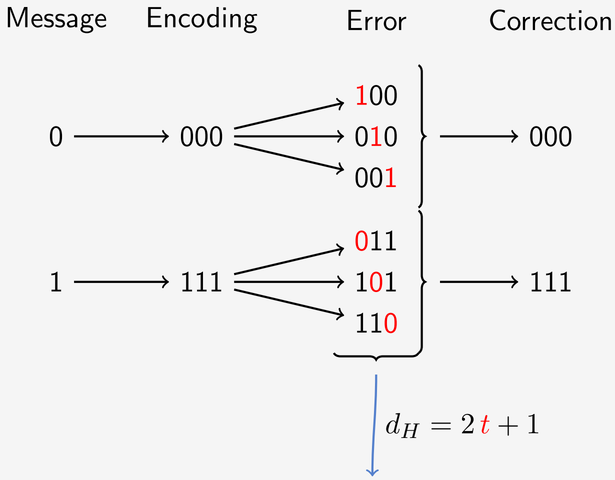

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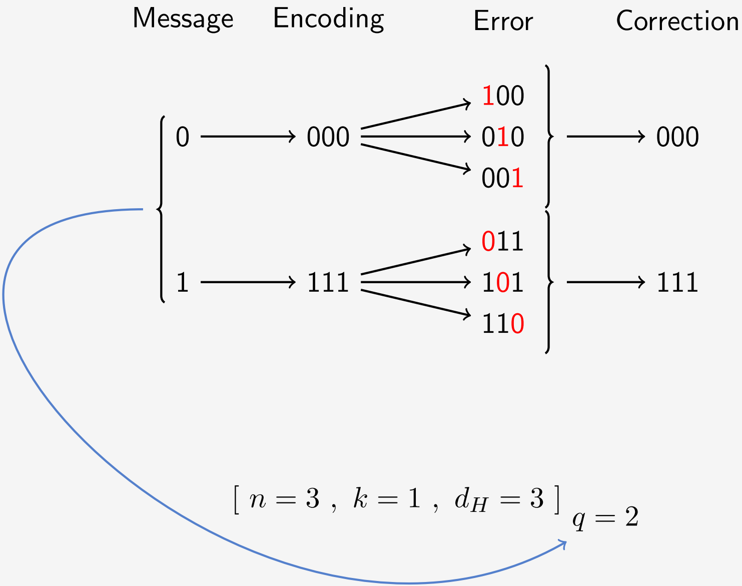

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Maximal distance separable (MDS) codes

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Is this optimal?

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Maximal distance separable (MDS) codes

$$[\, n = 3 \,,\ k = 1 \,,\ d_H = 3 \,]_{\,q = 2}$$

Is this optimal? Yes!

Singleton bound [7]

$$d_H \leq n - k + 1$$

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Constructing linear MDS codes

Message Generator matrix Code word

$$m \quad G_{k\times n} \;=\; c$$

Constructing linear MDS codes

Message, Generator matrix, Code word

$$m \; G_{k\times n} = c$$

This only makes sense if you can take linear combinations of messages and code words!

Yes. Right. Solution: Finite fields

Integers modulo q for q prime are a finite field.

Constructing linear MDS codes

Message → m, Generator matrix → $G_{k\times n}$, Code word → c

$$m \quad G_{k\times n} \; = \; c$$

G Has standard form (by taking linear combinations of code words)

$$G_{k\times n} = [\mathbb{1}_k | A]$$

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Smallest Hamming dist. given by smallest dist. to all zero code word. $\Longrightarrow$ Code is MDS iff any subset of k columns of $G_{k\times n}$ is linearly independent. $\Longleftrightarrow$ All square sub-matrices of A are non-singular.

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- Consider $\vec{v}, \vec{w} \in [q]^{\lfloor n/2 \rfloor}$, then the product states

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\begin{aligned}
&|\vec{v}\ G_{\lfloor n/2 \rfloor \times n}\rangle \\
\text{and} \quad &|\vec{w}\, G_{\lfloor n/2 \rfloor \times n}\rangle
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remain orthogonal upon tracing out up to $\lceil n/2 \rceil$ subsystems.

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  remain orthogonal upon tracing out up to $\lceil n/2 \rceil$ subsystems.
- Hence, we can construct a minimal support AME state like this:

$$|\Psi\rangle \propto \sum_{\vec{v} \in [q]^{\lfloor n/2 \rfloor}} |\vec{v}\, G_{k \times n}\rangle$$

An example

Generator matrix of a $[6,3,4]_5$ MDS code:

$$G_{3\times 6} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{array}\right].$$

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Yields minimal support AME state for $n = 6$, $q = 5$:

$$|\Psi\rangle = \sum_{\vec{v}\in GF(5)^3} |\vec{v}G\rangle = \sum_{i,j,l=0}^{4} |i, j, l, i+j+l, i+2j+3l, i+3j+4l\rangle$$

Can construct such states for all $n \leq q-1$ and q prime [arXiv:1701.03359].

Stabilizers of $|\Psi\rangle$ can be read off from the the G matrix.

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unitary

$$|\psi\rangle \in (\mathbb{C}^q)^{\otimes k} \longrightarrow |\varphi\rangle \in \mathcal{C} \subset (\mathbb{C}^q)^{\otimes n}$$

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Message Encoding Error Correction

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Quantum singleton bound [10–12]

$$2\,t + 1 = d \leq \frac{n-k}{2} + 1$$

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QECCs from minimal support AME states

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- We can explicitly construct the codes up to $n = 8$.
- The conjecture is true if there exist certain operators M whose weight cannot be decreased by multiplying it with stabilizers of $|\Psi\rangle$ ... and we can almost show that such M exist for all even n.

The code

Given a minimal support AME state $|\Psi\rangle$:

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- Claim: If M is an incompressible Pauli string operator, then $\mathcal{C}$ is a $[[n, 1, n/2]]_q$ QECC.

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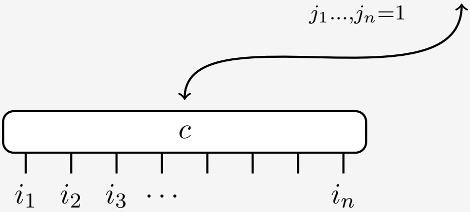

"perfect tensor"

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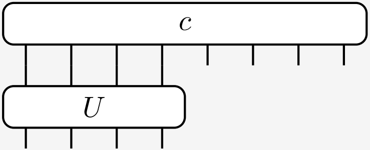

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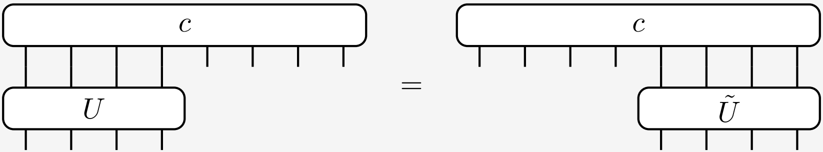

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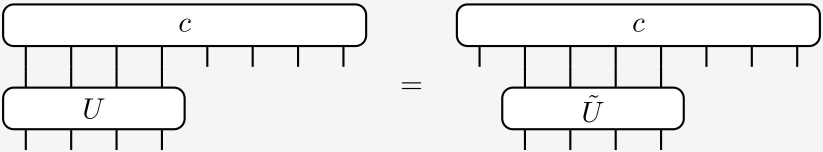

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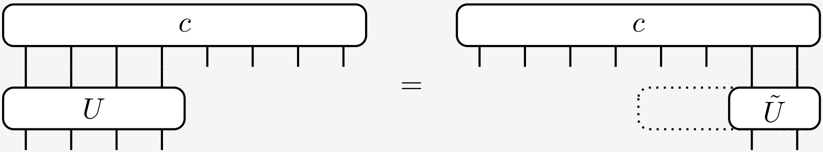

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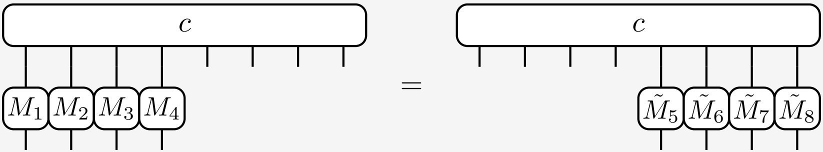

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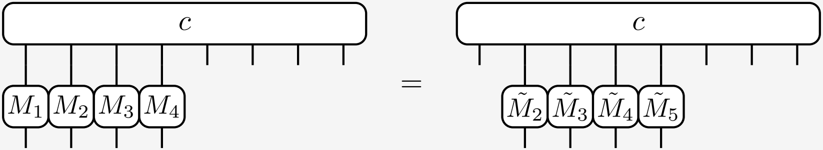

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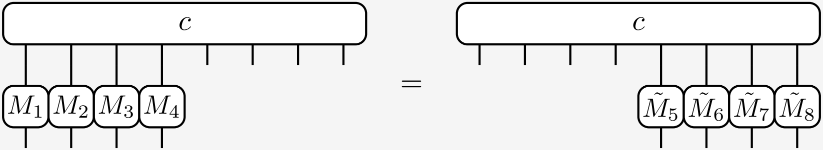

$\mathrm{Tr}(M_i) = 0$

Some intuition

- Remember EPR state: $(U \otimes \mathbb{1})|\psi^+\rangle = (\mathbb{1} \otimes U^\dagger)|\psi^+\rangle$
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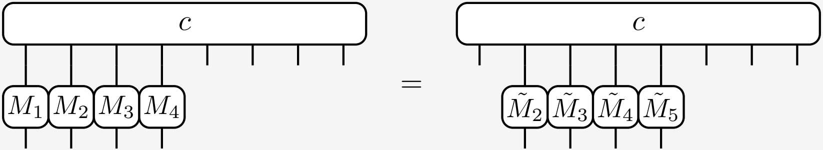

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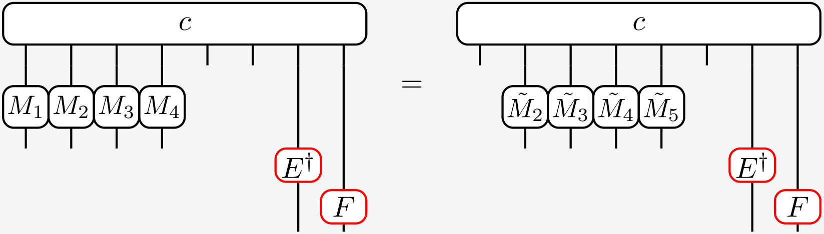

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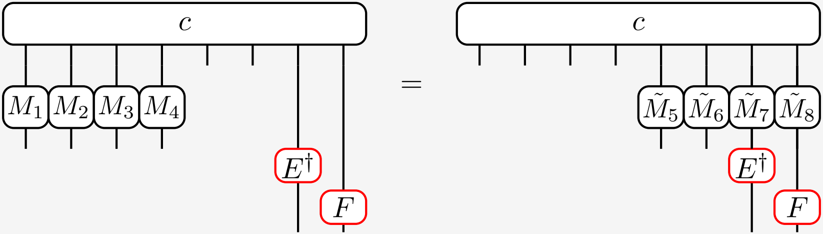

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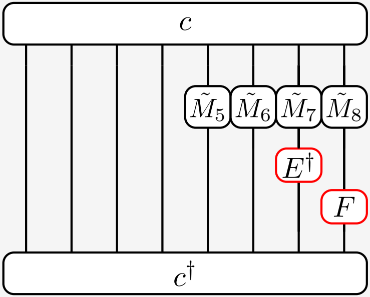
$$0 \stackrel{!}{=} \langle \Psi_1 | E^\dagger \, F | \Psi_0 \rangle =$$



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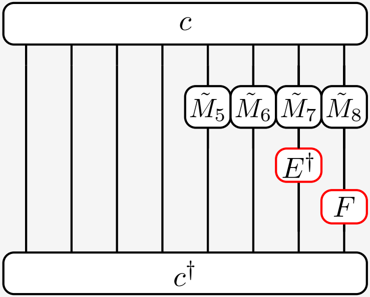
$$\mathrm{Tr}(\tilde{M}_5) \; \mathrm{Tr}(\tilde{M}_{6,7,8} \, E^\dagger \, F) =$$



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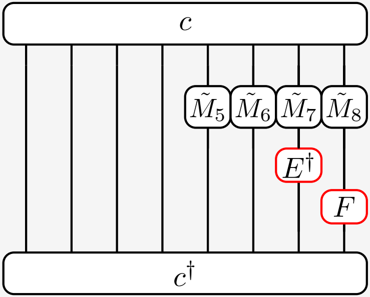
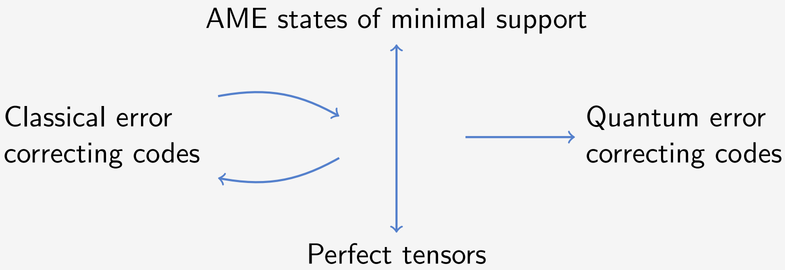
✓ $0 = \mathrm{Tr}(\tilde{M}_5)\ \mathrm{Tr}(\tilde{M}_{6,7,8}\, E^\dagger\, F) =$

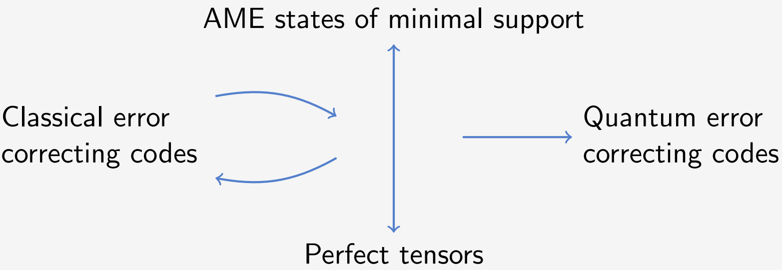


Summary


AME states of minimal support
Classical error correcting codes
Quantum error correcting codes
Perfect tensors

Summary

Thank you for your attention!



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