EFFICIENT UNITARY DESIGNS
WITH NEARLY TIME-INDEPENDENT HAMILTONIAN DYNAMICS

YOSHIFUMI NAKATA

UNIVERSITAT AUTONOMA DE BARCELONA (UAB)

THE UNIVERSITY OF TOKYO (UT)

WITH C. HIRCHE (UAB), M. KOASHI (UT), AND A. WINTER(UAB)

PRX 7, 021006 (2017) (SEE ALSO ARXIV:1609.07021)

OUTLINE

1. Introduction
   - Why care about random unitaries?
   - Main results
2. Constructions of designs
   - By Q. circuits
   - By spin-glass Hamiltonians
3. Conclusion and outlook

WHY RANDOM UNITARIES?

Random unitaries are very useful in QIP and in fundamental physics.

WHY RANDOM UNITARIES?

Random unitaries are very useful in QIP and in fundamental physics.

Applications in QIP

- Quantum communication
  - Achieving Q. capacity [Hayden et al '07]
- Quantum computation
  - Q. computational supremacy [Boixo '16].
- Experiments of quantum devices
  - Randomized benchmarking test [Knill et.al '08]
- Q. encryption, Q. algorithms, etc…

WHY RANDOM UNITARIES?

Random unitaries are very useful in QIP and in fundamental physics.

Applications in QIP

- Quantum communication
  - Achieving Q. capacity [Hayden et al '07]
- Quantum computation
  - Q. computational supremacy [Boixo '16].
- Experiments of quantum devices
  - Randomized benchmarking test [Knill et.al '08]
- Q. encryption, Q. algorithms, etc…

A standard tool in QIP

WHY RANDOM UNITARIES?

Random unitaries are very useful in QIP and in fundamental physics.

Applications in QIP

- Quantum communication
  - Achieving Q. capacity [Hayden et al '07]
- Quantum computation
  - Q. computational supremacy [Boixo '16].
- Experiments of quantum devices
  - Randomized benchmarking test [Knill et.al '08]
- Q. encryption, Q. algorithms, etc…

A standard tool in QIP

In fundamental physics

- Thermalization in isolated Q. systems
  - Pre-thermalization [Reimann '16]
- Quantum black holes
  - "Scrambling" [Hayden&Preskill '07]
- Quantum chaos
  - Out-of-time-ordered (OTO) correlators [Roberts&Yoshida '16]

WHY RANDOM UNITARIES?

Random unitaries are very useful in QIP and in fundamental physics.

Applications in QIP

- Quantum communication
  - Achieving Q. capacity [Hayden et al '07]
- Quantum computation
  - Q. computational supremacy [Boixo '16].
- Experiments of quantum devices
  - Randomized benchmarking test [Knill et.al '08]
- Q. encryption, Q. algorithms, etc…

A standard tool in QIP

In fundamental physics

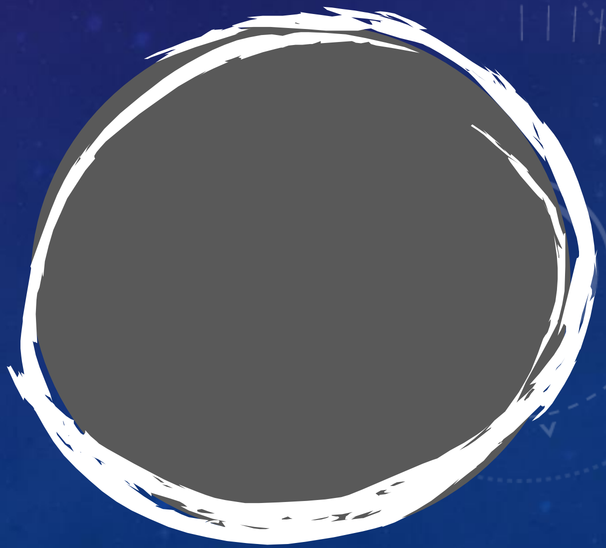
- Thermalization in isolated Q. systems
  - Pre-thermalization [Reimann '16]
- Quantum black holes
  - "Scrambling" [Hayden&Preskill '07]
- Quantum chaos
  - Out-of-time-ordered (OTO) correlators [Roberts&Yoshida '16]

A key to understanding complex Q. systems

WHAT RANDOM UNITARIES?

- Haar random unitary = uniformly distributed random unitary

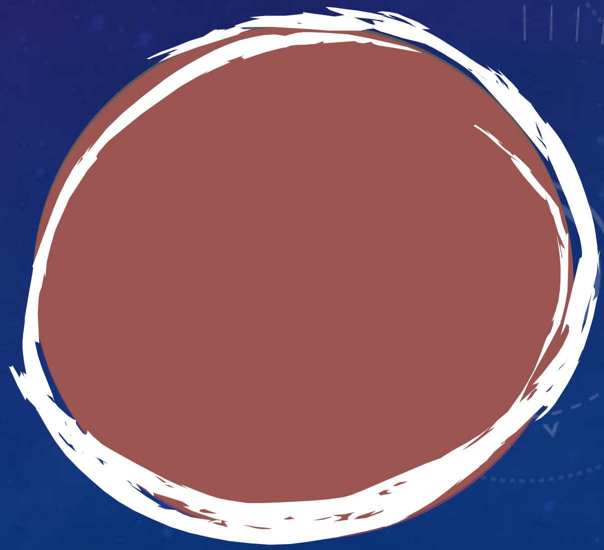
Unitary group



WHAT RANDOM UNITARIES?

- Haar random unitary = uniformly distributed random unitary

Unitary group

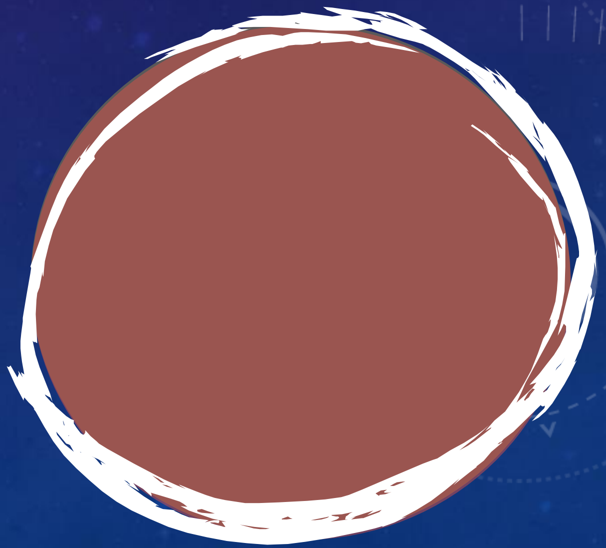


A distribution of
a Haar random unitary

WHAT RANDOM UNITARIES?

- Haar random unitary = uniformly distributed random unitary
  - On N qubits, it takes at least in time $O(2^{2^N})$.
  - Need to think approximate versions.

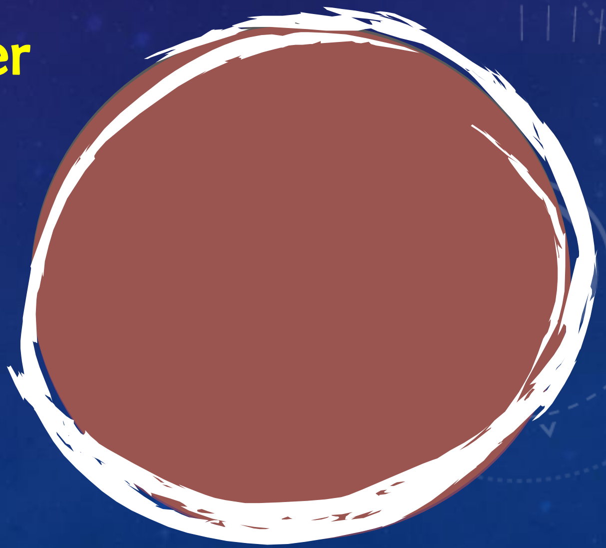
Unitary group



A distribution of
a Haar random unitary

WHAT RANDOM UNITARIES?

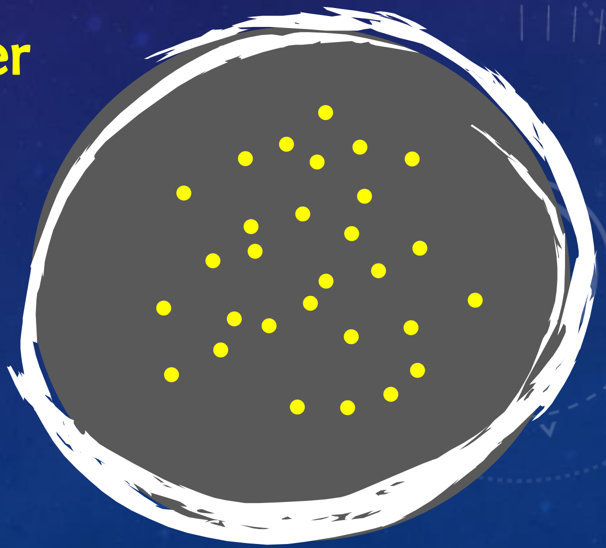
- Haar random unitary = uniformly distributed random unitary
  - On N qubits, it takes at least in time $\boldsymbol{O}(2^{2^N})$.
  - Need to think approximate versions.
- An ϵ-approximate unitary t-design simulates up to the tth order statistical moments of a Haar random unitary within an error ϵ.
  - E.g.) 1-designs (2-designs) for averages (variances)



WHAT RANDOM UNITARIES?

- Haar random unitary = uniformly distributed random unitary
  - On N qubits, it takes at least in time $O(2^{2^N})$.
  - Need to think approximate versions.
- An ϵ-approximate unitary t-design simulates up to the tth order statistical moments of a Haar random unitary within an error ϵ.
  - E.g.) 1-designs (2-designs) for averages (variances)

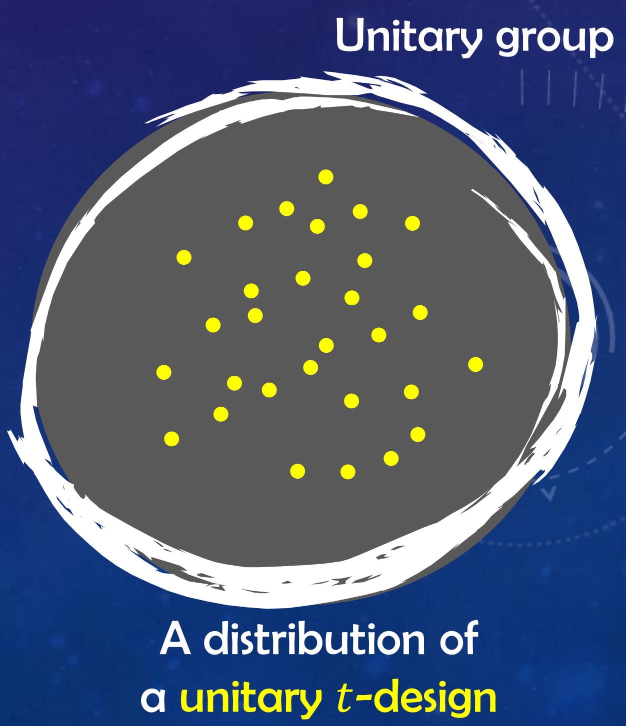
Unitary group



A distribution of a unitary t-design

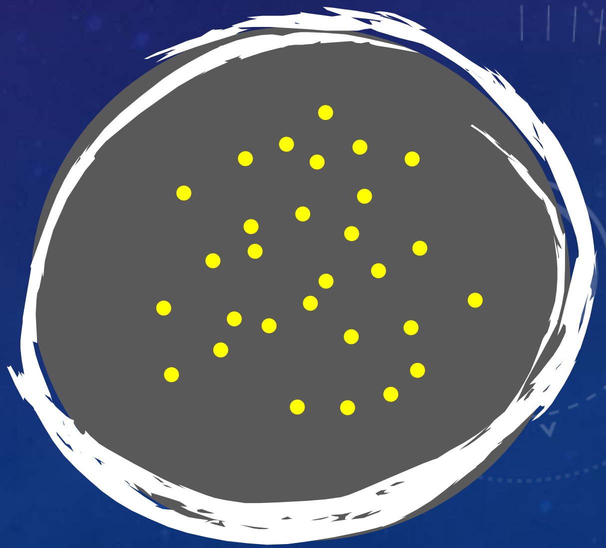
VARIOUS RESULTS ON DESIGNS

- In QIP, unitary designs are as useful as Haar random unitaries.



VARIOUS RESULTS ON DESIGNS

- In QIP, unitary designs are as useful as Haar random unitaries.
  - 2-designs suffice in most applications in QIP [Low '09].
  - Some protocols require 3 or 4-designs.
    - Query complexity [Brandao&Horodecki'13], sensing [Kueng et al '14]
  - In general, higher-designs = better performance
    - Due to the large deviation bounds [Low '09]



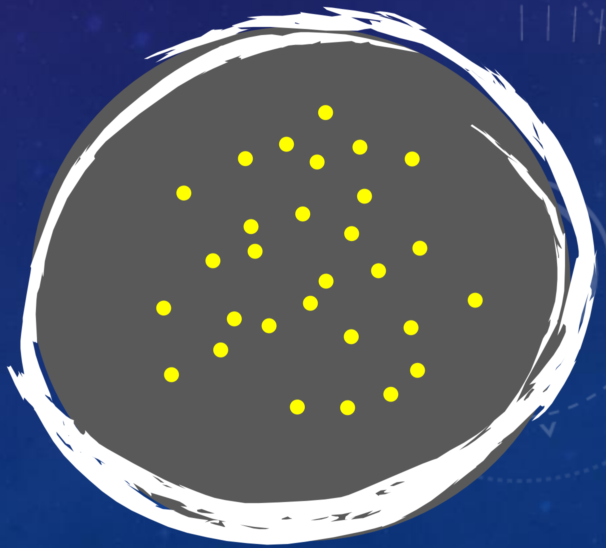
A distribution of
a unitary t-design

VARIOUS RESULTS ON DESIGNS

- In QIP, unitary designs are as useful as Haar random unitaries.
  - 2-designs suffice in most applications in QIP [Low '09].
  - Some protocols require 3 or 4-designs.
    - Query complexity [Brandao&Horodecki'13], sensing [Kueng et al '14]
  - In general, higher-designs = better performance
    - Due to the large deviation bounds [Low '09]

Goal: to generate unitary designs by Q. circuits as efficiently as possible.

Unitary group



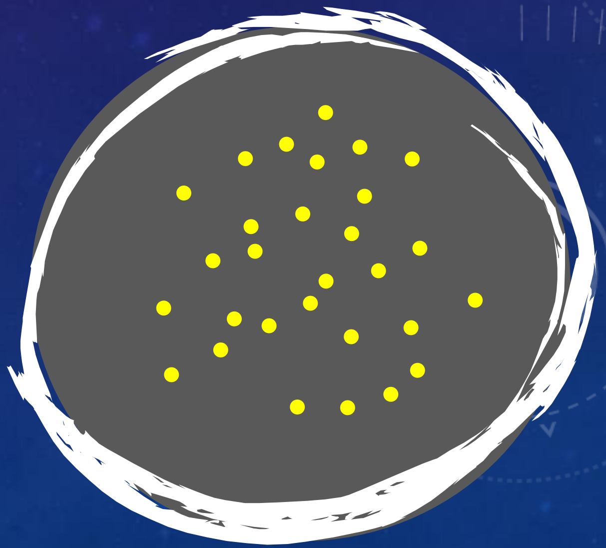
A distribution of a unitary t-design

VARIOUS RESULTS ON DESIGNS

- In fundamental physics, designs are used to quantify the "complexity" of dynamics.
  - Thermal relaxation in isolated systems [Low '09]
  - Quantifying quantum chaos [Roberts&Yoshida '16]
  - If a Hamiltonian quickly generates designs, it may be a "dual" to quantum gravity [Kitaev '16?].

Talk by Zi-Wen Liu this morning

Unitary group



A distribution of
a unitary t-design

VARIOUS RESULTS ON DESIGNS

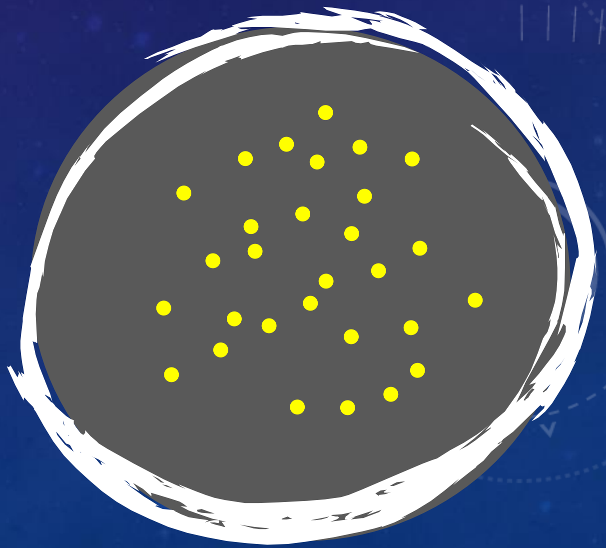
- In fundamental physics, designs are used to quantify the “complexity” of dynamics.
  - Thermal relaxation in isolated systems [Low ‘09]
  - Quantifying quantum chaos [Roberts&Yoshida ‘16]
  - If a Hamiltonian quickly generates designs, it may be a “dual” to quantum gravity [Kitaev ‘16?].

Talk by Zi-Wen Liu this morning

Goal: to find a “natural” many-body Hamiltonian that generates unitary designs as quickly as possible.

E.g.) Heisenberg interaction, time-independent, local interactions.....

Unitary group



A distribution of a unitary t-design

TWO GOALS WHEN GENERATING DESIGNS

Goal: to generate unitary designs by Q. circuits as efficiently as possible.

Goal: to find a "natural" many-body Hamiltonian that generates unitary designs as quickly as possible.

In this talk, we especially consider $N \gg 1, t = O(1)$.

WHAT ARE KNOWN?

- Two constructions for t-designs ($t \geq 3$) on N qubits

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	Methods	QFT + classical TPE		
	# of gates	$O(t^3 N^3)$ (Brodsky&Hoory '13)		
	Works for	$t = O(N/\log N)$		
By Hamiltonian dynamics ($N \gg 1$)	Hamiltonian	Not natural		
	Time-dependence	N.A.		

WHAT ARE KNOWN?

- Two constructions for t-designs ($t \geq 3$) on N qubits

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	Methods	QFT + classical TPE	Local random circuits	
	# of gates	$O(t^3N^3)$ (Brodsky&Hoory '13)	$O(t^{10}N^2)$	
	Works for	$t = O(N/\log N)$	$t = \mathrm{poly}\, N$	
By Hamiltonian dynamics ($N \gg 1$)	Hamiltonian	Not natural	Nearest-neighbor random interactions	
	Time-dependence	N.A.	Interactions need to be changed $O(t^{10}N)$ times.	

WHAT ARE KNOWN?

- Two constructions for t-designs ($t \geq 3$) on N qubits

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	Methods	QFT + classical TPE	Local random circuits	Hadamard + random diagonal circuits
	# of gates	$O(t^3N^3)$ (Brodsky&Hoory '13)	$O(t^{10}N^2)$	$O(tN^2)$
	Works for	$t = O(N/\log N)$	$t = \text{poly } N$	$t = O(\sqrt{N})$
By Hamiltonian dynamics ($N \gg 1$)	Hamiltonian	Not natural	Nearest-neighbor random interactions	
	Time-dependence	N.A.	Interactions need to be changed $O(t^{10}N)$ times.	

WHAT ARE KNOWN?

- Two constructions for t-designs ($t \geq 3$) on N qubits

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	Methods	QFT + classical TPE	Local random circuits	Hadamard + random diagonal circuits
	# of gates	$O(t^3N^3)$ (Brodsky&Hoory '13)	$O(t^{10}N^2)$	$O(tN^2)$
	Works for	$t = O(N/\log N)$	$t = \mathrm{poly}\, N$	$t = O(\sqrt{N})$
By Hamiltonian dynamics ($N \gg 1$)	Hamiltonian	Not natural	Nearest-neighbor random interactions	All-to-all spin-glass interactions
	Time-dependence	N.A.	Interactions need to be changed $O(t^{10}N)$ times.	Interactions need to be changed $O(t)$ times.

Our contribution is:

1. To construct more efficient q. circuits for designs.
2. To provide relatively natural Hamiltonians generating designs.

WHAT ARE KNOWN?

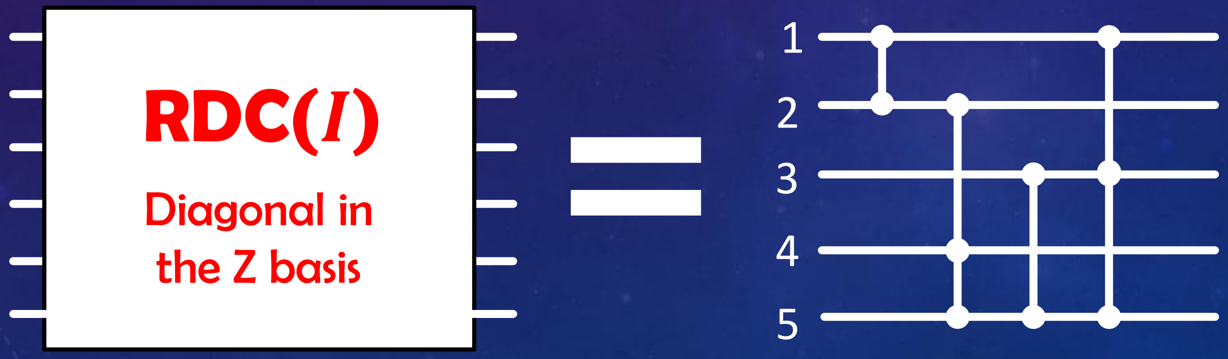
- Two constructions for t-designs ($t \geq 3$) on N qubits

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	Methods	QFT + classical TPE	Local random circuits	Hadamard + random diagonal circuits
	# of gates	$O(t^3N^3)$ (Brodsky&Hoory '13)	$O(t^{10}N^2)$	$O(tN^2)$
	Works for	$t = O(N/\log N)$	$t = \mathrm{poly}\, N$	$t = O(\sqrt{N})$
By Hamiltonian dynamics ($N \gg 1$)	Hamiltonian	Not natural	Nearest-neighbor random interactions	All-to-all spin-glass interactions
	Time-dependence	N.A.	Interactions need to be changed $O(t^{10}N)$ times.	Interactions need to be changed $O(t)$ times.

2. NEW CONSTRUCTIONS OF DESIGNS

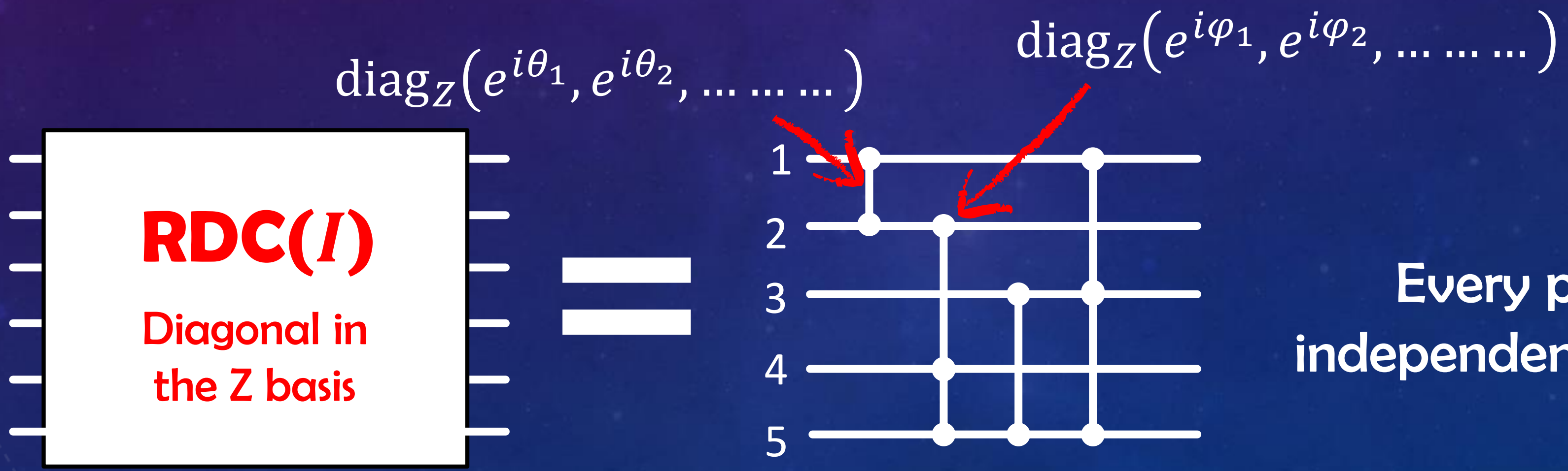
Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

- Random diagonal circuits (RDC(I))
  - The I (a set of subsets in {1,…,N}) specifies where random diagonal gates are applied.



Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

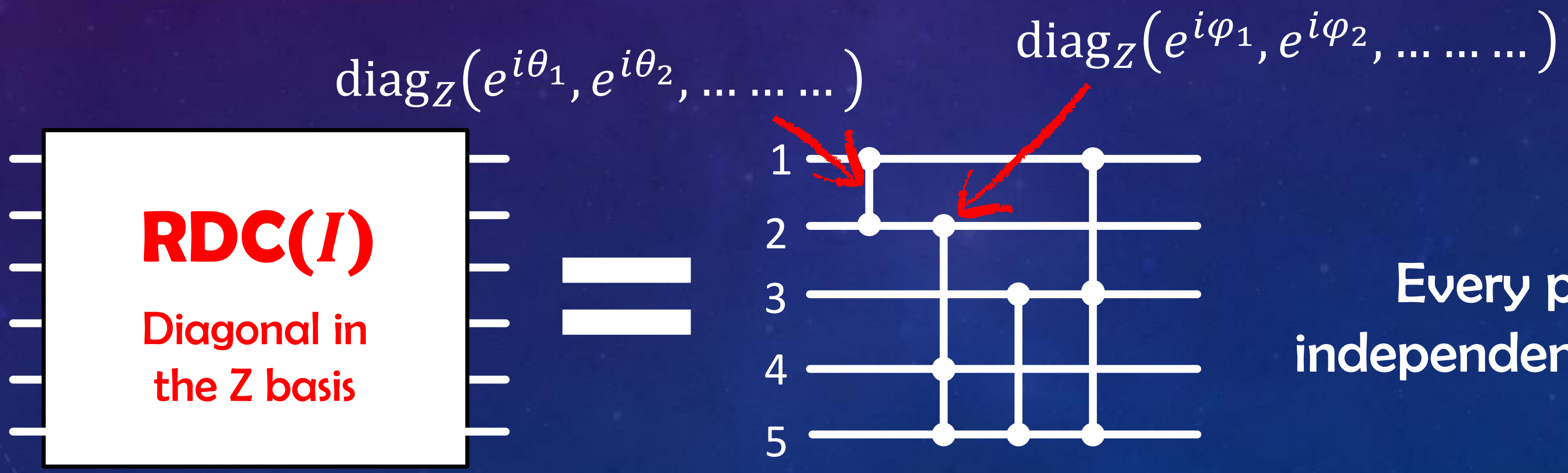
- Random diagonal circuits (RDC(I))
  - The I (a set of subsets in {1,...,N}) specifies where random diagonal gates are applied.



Every phase is randomly & independently chosen from $[0, 2\pi)$.

Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

- Random diagonal circuits (RDC(I))
  - The I (a set of subsets in {1,...,N}) specifies where random diagonal gates are applied.

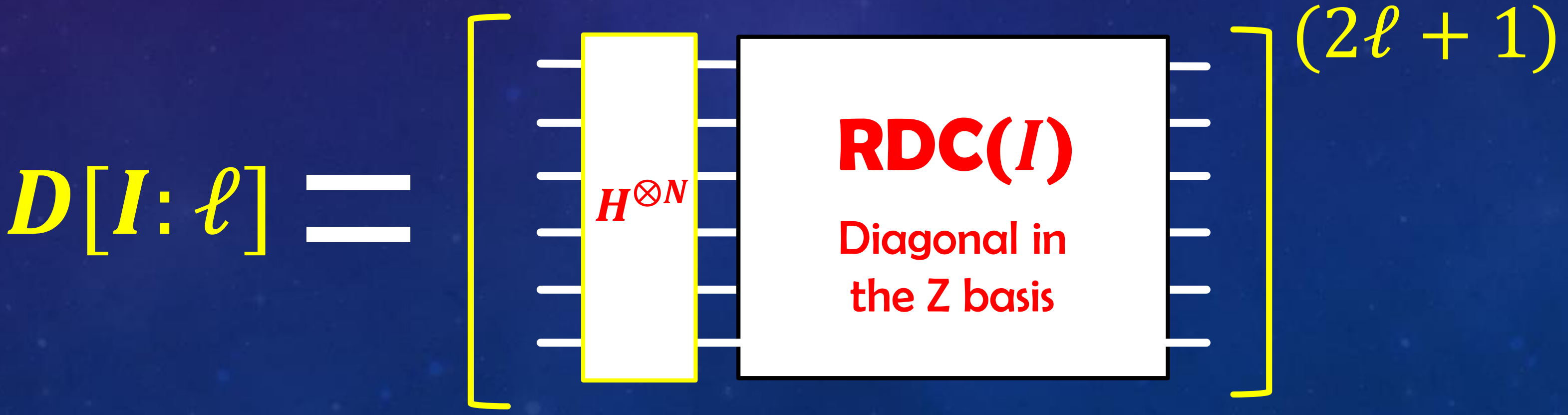


Every phase is randomly & independently chosen from $[0, 2\pi)$.

E.g.) $\mathrm{I} = \{I_1, I_2, I_3, I_4\}$ $(I_1 = \{1,2\}, I_2 = \{2,4,5\}, I_3 = \{3,5\}, I_4 = \{1,3,5\})$

Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

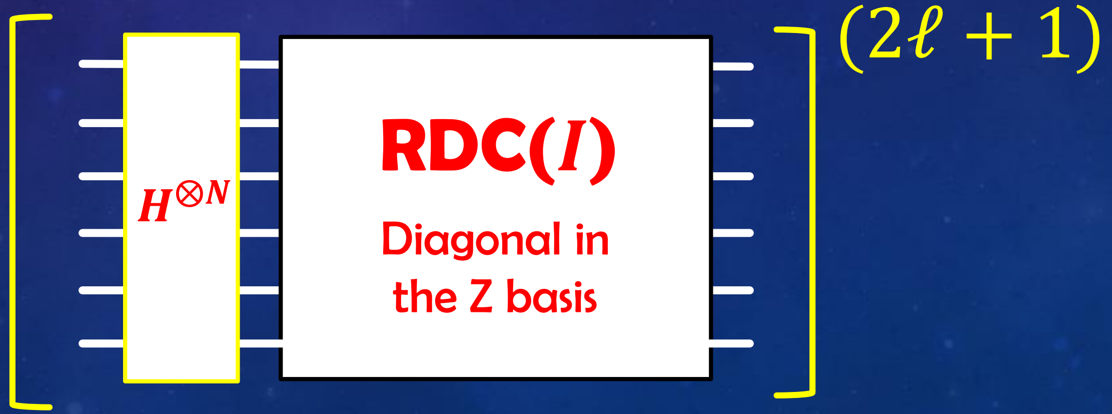
- Repeat RDC(I) and the Hadamard gates $H^{\otimes N}$, $(2\ell + 1)$ times.



Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

- Repeat RDC(I) and the Hadamard gates $H^{\otimes N}$, $(2\ell + 1)$ times.
  - Random rotations around two complementary axes, Z-axis and X-axis.
  - Natural to expect that $\boldsymbol{D}[\boldsymbol{I}:\ell]$ eventually bcomes a unitary design.... but for what I?

$\boldsymbol{D}[\boldsymbol{I}:\ell] =$



Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

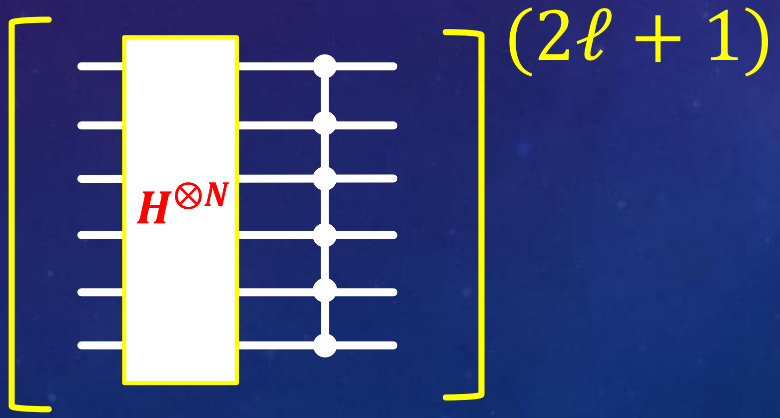
- Two choices of I:
  1. $I_{\mathrm{all}} = \{1,2, \dots\dots, N\}$: "global" random diagonal unitary (not efficient).
  2. $I_2 = \{(i,j) : i \neq j\}$: random diagonal gates on all pairs of 2 qubits.

Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

- Two choices of I:
    1. $I_{\mathrm{all}} = \{1,2,\ldots\ldots,N\}$: "global" random diagonal unitary (not efficient).
    2. $I_2 = \{(i,j): i \neq j\}$: random diagonal gates on all pairs of 2 qubits.

1. $I_{\mathrm{all}} = \{1,2,\ldots\ldots,N\}$

$D[I_{\mathrm{all}}:\ell] =$

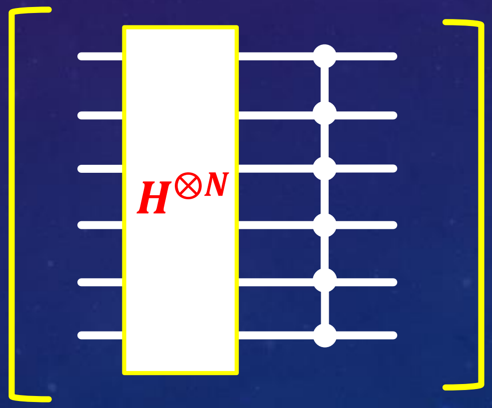


Q. CIRCUITS FOR DESIGNS –MAIN IDEA–

- Two choices of I:
    1. $I_{\text{all}} = \{1,2,\ldots\ldots,N\}$: "global" random diagonal unitary (not efficient).
    2. $I_2 = \{(i,j): i \neq j\}$: random diagonal gates on all pairs of 2 qubits.

1. $I_{\text{all}} = \{1,2,\ldots\ldots,N\}$

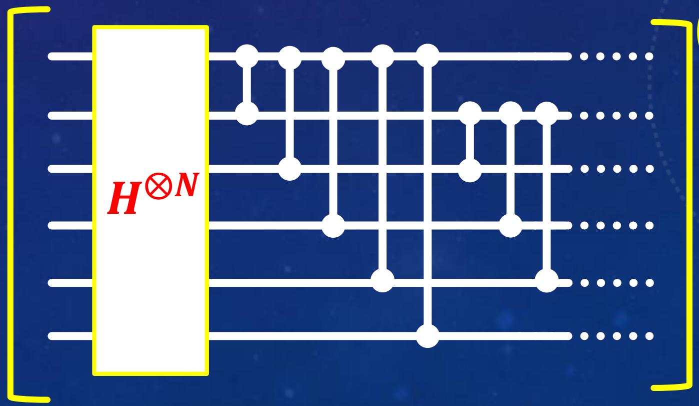
$D[I_{\text{all}}:\ell] =$



$(2\ell+1)$

2. $I_2 = \{(i,j): i \neq j\}$

$D[I_2:\ell] =$

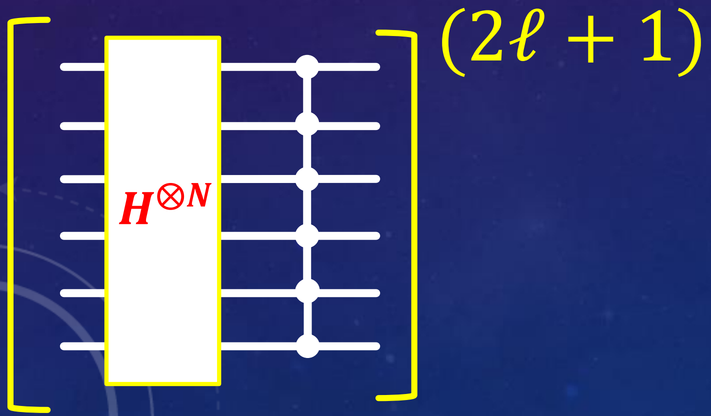


$(2\ell+1)$

Q. CIRCUITS FOR DESIGNS –THEOREM 1-

<u>Theorem 1</u>
Suppose $t = O(N/\log N)$.
Then, $D[I_{\text{all}}: \ell]$ is an ϵ-approximate unitary t-design if $\ell \geq O(t + \log 1/\epsilon)$.

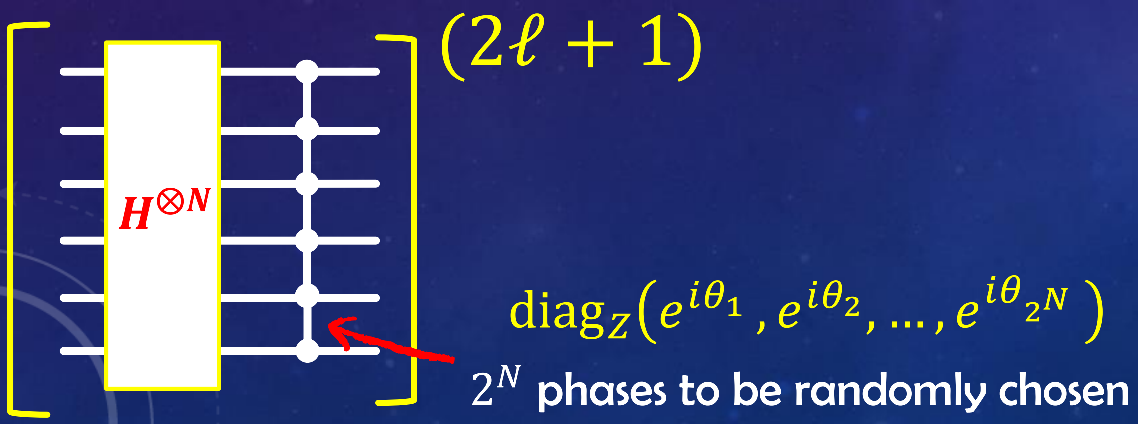
- Proof by a quantum tensor-product expander [Hastings&Harrow'08].



Q. CIRCUITS FOR DESIGNS –THEOREM 1-

<u>Theorem 1</u>
Suppose $t = O(N/\log N)$.
Then, $D[I_{\text{all}}: \ell]$ is an ϵ-approximate unitary t-design if $\ell \geq O(t + \log 1/\epsilon)$.

- Proof by a quantum tensor-product expander [Hastings&Harrow'08].
- But, *not efficient.*

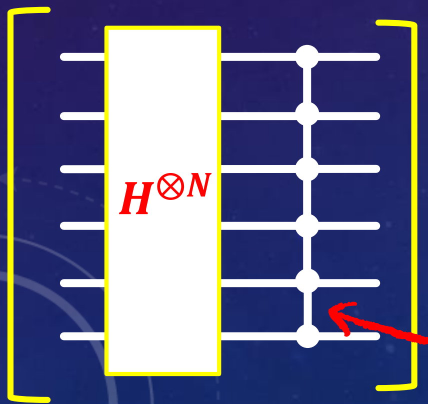


Q. CIRCUITS FOR DESIGNS –THEOREM 1-

<u>Theorem 1</u>
Suppose $t = O(N/\log N)$.
Then, $D[I_{\mathrm{all}}\colon \ell]$ is an ϵ-approximate unitary t-design if $\ell \geq O(t + \log 1/\epsilon)$.

- Proof by a quantum tensor-product expander [Hastings&Harrow'08].
- But, *not efficient.*

If we can **efficiently simulate** $\mathrm{diag}_Z\left(e^{i\theta_1}, \dots, e^{i\theta_{2^N}}\right)$, **up to the tth order,** we obtain efficient unitary t-designs directly from Theorem 1 .



$(2\ell + 1)$

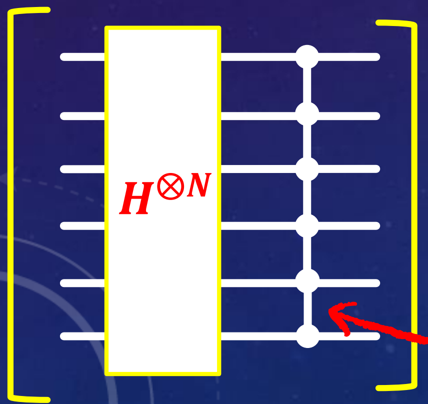
$\mathrm{diag}_Z\left(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{2^N}}\right)$

2^N phases to be randomly chosen

Q. CIRCUITS FOR DESIGNS –THEOREM 1-

Theorem 1
Suppose $t = O(N/\log N)$.
Then, $D[I_{\text{all}}: \ell]$ is an ϵ-approximate unitary t-design if $\ell \geq O(t + \log 1/\epsilon)$.

- Proof by a quantum tensor-product expander [Hastings&Harrow'08].
- But, *not efficient.*



$(2\ell + 1)$

$\mathrm{diag}_Z\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{2^N}}\right)$

2^N phases to be randomly chosen

If we can efficiently simulate $\mathrm{diag}_Z\left(e^{i\theta_1}, \ldots, e^{i\theta_{2^N}}\right)$, up to the tth order, we obtain efficient unitary t-designs directly from Theorem 1 .

If $t = O(\sqrt{N})$,

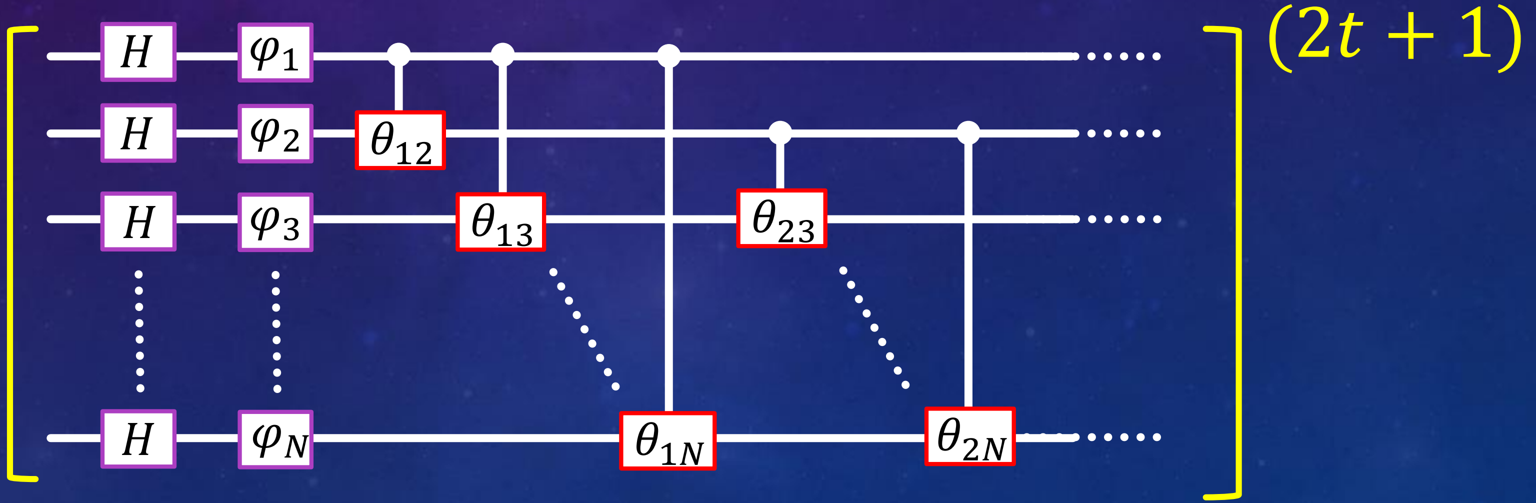
$$D[I_{\text{all}}: \ell] \approx D[I_2: \ell]$$

up to the tth order

Q. CIRCUITS FOR DESIGNS –THEOREM 2-

Theorem 2

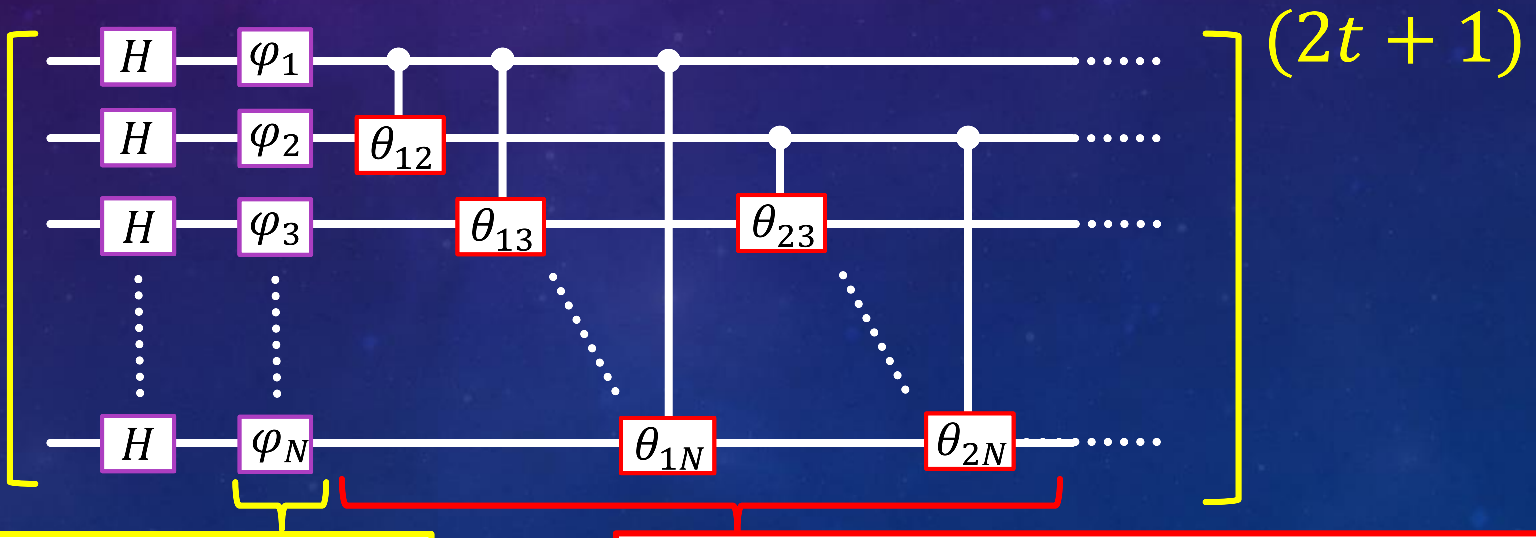
Suppose $t = O(\sqrt{N})$. Then, the Q. circuit below is an ϵ-approximate unitary t-design with $\Theta(N(tN + \log_2(1/\epsilon)))$ gates.



Q. CIRCUITS FOR DESIGNS –THEOREM 2-

Theorem 2

Suppose $t = O(\sqrt{N})$. Then, the Q. circuit below is an ϵ-approximate unitary t-design with $\Theta(N(tN + \log_2(1/\epsilon)))$ gates.



$\mathrm{diag}_Z\left(1, e^{i\varphi_k}\right)$ $\left(\varphi_k \in \{0, \frac{2\pi}{t+1}, \dots, \frac{2t\pi}{t+1}\}\right)$

$\mathrm{diag}_Z(1,1,1,e^{i\theta_{k\ell}})$ $\left(\theta_{k\ell} \in \{0, \frac{2\pi}{\lfloor t/2 \rfloor+1}, \frac{4\pi}{\lfloor t/2 \rfloor+1}, \dots, \frac{2t\pi}{\lfloor t/2 \rfloor+1}\}\right)$.

DESIGN HAMILTONIAN

Design Hamiltonian

A t-**design Hamiltonian** is a family of Hamiltonian $\{H_\mu\}_\mu$ such that the time-evolving operators $\{e^{-iH_\mu T}\}_\mu$ is an ϵ-approximate unitary t-design for any time $T \geq T_{\mathrm{th}}$.

DESIGN HAMILTONIAN

Design Hamiltonian

A t-design Hamiltonian is a family of Hamiltonian $\{H_\mu\}_\mu$ such that the time-evolving operators $\{e^{-iH_\mu T}\}_\mu$ is an ϵ-approximate unitary t-design for any time $T \geq T_{\text{th}}$.

Corollary 3

Let $H_{XZ}(T)$ be a time-dependent Sherrington-Kirkpatrick spin-glass Hamiltonian,

$$H_{XZ}(T) = \begin{cases} -\sum_{i<j} J_{ij}\sigma_i^Z\sigma_j^Z - \sum_i B_i\sigma_i^Z & \text{if } \ 2m\pi \leq T < (2m+1)\pi \\ -\sum_{i<j} J_{ij}\sigma_i^X\sigma_j^X - \sum_i B_i\sigma_i^X & \text{if } \ (2m+1)\pi \leq T < 2(m+1)\pi \end{cases}$$

Then, the $H_{XZ}(T)$ is a t-design Hamiltonian where $T_{\text{th}} \approx 2\pi t$.

DESIGN HAMILTONIAN

- The $H_{XZ}(T)$ is a t-design Hamiltonian with $T_{\mathrm{th}} \approx 2\pi t$.
  - "Quick" generation of designs.

DESIGN HAMILTONIAN

- The $H_{XZ}(T)$ is a t-design Hamiltonian with $T_{\mathrm{th}} \approx 2\pi t$.
  - "Quick" generation of designs.
  - Time-dependent and all-to-all interactions.

DESIGN HAMILTONIAN

- The $H_{XZ}(T)$ is a t-design Hamiltonian with $T_{\mathrm{th}} \approx 2\pi t$.
  - "Quick" generation of designs.
  - Time-dependent and all-to-all interactions.

<u>Natural design Hamiltonian conjecture</u>

∃ time-independent design Hamiltonians with local interactions, where $T_{\mathrm{th}} = O(t \log N)$.

DESIGN HAMILTONIAN

- The $H_{XZ}(T)$ is a t-design Hamiltonian with $T_{\mathrm{th}} \approx 2\pi t$.
  - "Quick" generation of designs.
  - Time-dependent and all-to-all interactions.

Natural design Hamiltonian conjecture

∃ time-independent design Hamiltonians with local interactions, where $T_{\mathrm{th}} = O(t \log N)$.

- Generalizing the "fast scrambling conjecture" [Sekino&Susskind '08].
  - Q. black holes are 2-design Hamiltonian with $T_{\mathrm{th}} = O(\log N)$.
  - ∀ time-independent 2-design Hamiltonians with local interactions, $T_{\mathrm{th}} \geq O(\log N)$.

3. CONCLUSION AND OUTLOOK

CONCLUSION

- New efficient implementations of unitary designs.
  - By Q. circuits with $O(tN^2)$ gates
  - By the dynamics with weakly time-dependent spin-glass Hamiltonian.

		Harrow and Low '09	Brandao et al '13	Our result
By Q. circuits	# of gates	$O(t^2N^2(tN + \log 1/\epsilon))$	$O(t^9N(tN + \log 1/\epsilon))$	$O(N(tN + \log 1/\epsilon))$
	Works for	$t = O(N/\log N)$	$t = O(\mathrm{poly}\, N)$	$t = O(\sqrt{N})$
By Hamiltonian dynamics	Hamiltonian	Not natural	NN interactions	All-to-all spin-glass
	Time-dependence	N.A.	Strong	Weak

- Proposed a design Hamiltonian.

OUTLOOK

- Lower bound on the # of gates to achieve unitary t-design?
  - Trivial bound is $O(tN)$, even approximately.
  - Fast scrambling conjecture claims it should be $O(N \log N)$ for 2-designs.

OUTLOOK

- Lower bound on the # of gates to achieve unitary t-design?
  - Trivial bound is $O(tN)$, even approximately.
  - Fast scrambling conjecture claims it should be $O(N \log N)$ for 2-designs.
- Natural design Hamiltonian conjecture
  - New Q. spin-glasses or Q. chaos, which may be dual to Q. black holes.
  - The Sachdev-Ye-Kitaeve Hamiltonian (all-to-all interactions)?

OUTLOOK

- Lower bound on the # of gates to achieve unitary t-design?
  - Trivial bound is $O(tN)$, even approximately.
  - Fast scrambling conjecture claims it should be $O(N \log N)$ for 2-designs.
- Natural design Hamiltonian conjecture
  - New Q. spin-glasses or Q. chaos, which may be dual to Q. black holes.
  - The Sachdev-Ye-Kitaeve Hamiltonian (all-to-all interactions)?

THANK YOU FOR YOUR ATTENTION!

PRX 7, 021006 (2017) (SEE ALSO ARXIV:1609.07021)