Information Geometry of Dense Coding

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The dense coding and its variants, as well as the celebrated quantum channel coding theorem, concern distinguishability among finitely many alternatives, and the proper quantum statistical framework for dealing with finite alternatives is the hypothesis testing. There is another, essentially different, framework in quantum statistics, called the parameter estimation, in which one deals with continuously many alternatives. Among recent development in the latter framework is a quantum channel identification problem [1] [2], in which one seeks the best strategy of estimating an unknown quantum operation Γ acting on the set $\mathcal{S}(\mathcal{H})$ of quantum states on a Hilbert space $\mathcal{H}$. In particular, improvement of estimation performance through entanglement and a rather unexpected transitionlike behavior of the optimal estimation scheme were reported in [1] through an analysis of a depolarization channel $\Gamma : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2)$.

In this talk, we present the identification problem of a unitary channel $\Gamma_U : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2) : \rho \mapsto U\rho U^*$, $(U \in SU(2))$ through the extension $\Gamma_U \otimes I : \sigma \longmapsto (U \otimes I)\sigma(U \otimes I)^*$ on $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, or its restriction $\Gamma_U \otimes I : |\psi\rangle\langle\psi| \longmapsto |(U \otimes I)\psi\rangle\langle(U \otimes I)\psi|$ on the set $\partial_e\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ of pure states that is identified with the 3-dimensional complex projective space $\mathbb{C}P^3$. Obviously this is regarded as a continuous analogue of the dense coding. The main results are summarized as follows.

(i) The quantum Cramér-Rao lower bound for the family $\{(U \otimes I)\psi\}_{U \in SU(2)}$ of output states takes the minimum, and is achievable, if and only if ψ is a maximally entangled state.

(ii) The Riemannian manifold of output states (i.e., an $SU(2)$ orbit) equipped with the quantum Fisher metric is isometric to the 3-dimensional real projective space $\mathbb{R}P^3 \cong SU(2)/\{\pm I\}$ if and only if ψ is a maximally entangled state.

These results provide an alternative view for the optimality of the use of a maximally entangled state.

References

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