Estimation of Fidelity to Multiparticle Entangled State using the Maximum Likelihood Principle

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Recently, the importance of maximally entangled state of spin-1/2 particles (Bell state, GHZ state, etc.) has become perceived not only as it is a fundamental concept of quantum mechanics [1] but also as it is essential for quantum information processing [2]. There are several experimental reports on 2-, 3- and 4-particle entangled states [3]. We apply the maximum likelihood principle [4] to estimation of quantum state which is experimentally generated as a candidate of the entangled 2-, 3-, or n-particle state. We especially pay attention to observables which are included in the simplest expansion for the desired entangled state. There is a research about estimation quantum state by means of the maximum likelihood principle which uses all single measurement data [5]. Here, however, we deal with a different situation, namely, the case that only the observed mean value is available. In our method, we pay attention to the fidelity of the generated state to the desired entangled state instead of estimating the state itself.

At first, we introduce the *simple expansion* for the n-particle GHZ state, $|\Phi_n\rangle$, which is defined as

$$|\Phi_n\rangle := (1/\sqrt{2})(|+_1;+_2;\cdots;+_n\rangle + |-_1;-_2;\cdots;-_n\rangle). \quad (1)$$

This is written in the density operator form as

$$\begin{aligned} |\Phi_n\rangle\langle\Phi_n| &= \frac{1}{2^{n+1}}\left(\prod_{j=1}^{n}(\sigma_x^j + i\sigma_y^j) + \prod_{j=1}^{n}(\sigma_x^j - i\sigma_y^j)\right. \\ &\quad \left. + \prod_{j=1}^{n}(I^j + \sigma_z^j) + \prod_{j=1}^{n}(I^j - \sigma_z^j)\right) \\ &=: (1/2^n)(I + O_1 + O_2 + \cdots + O_{2^n-1}). \end{aligned} \quad (2)$$

Expansion of the right hand of Eq. (2) leads to a summation of $2^n - 1$ terms of product of the Pauli matrices and one $2^n \times 2^n$ identity operator. Hereafter, we refer to this type of expansion as the *simple expansion*. These operators have been represented by $I, O_1, O_2, \cdots, O_{2^n-1}$. These 2^n kinds of operator construct some finite commutative group that we refer to as n-Entanglement Fidelity Group ($=: \Lambda$). If all of the expectation values of these operators are given, we can uniquely determine the fidelity to $|\Phi_n\rangle$ ($=: f$). In the following, however, we consider the case that only a summation of some operators in the simple expansion is given.

We consider the case that $\mathcal{A}$ is a summation of operators $O_1, O_2, \cdots, O_m$ which include, at least, all elements of a system of generators for Λ, that is, $\mathcal{A} := O_1 + O_2 + \cdots + O_m$ ($2^n - 1 \geq m \geq n$). We assume that only the mean value $\langle\mathcal{A}\rangle$ is given experimentally. The maximum likelihood principle requires that the estimated fidelity (to $|\Phi_n\rangle$) should be the one that makes the variance, $\langle(\Delta\mathcal{A})^2\rangle$, minimal. The estimated fidelity is summarized as

$$f_{\rm est} = \begin{cases} \dfrac{\langle\mathcal{A}\rangle - r_2}{m - r_2} & m \geq \langle\mathcal{A}\rangle > r_2 \\ 0 & r_2 \geq \langle\mathcal{A}\rangle \geq r_s, \end{cases} \quad (3)$$

where we write the eigenvalues of $\mathcal{A}$ as $m,\ r_2,\ r_3, \cdots,$ and r_s, where $m > r_2 > r_3 > \cdots > r_s \geq -m$ ($s \leq 2^n$).

We also show that when such kind of the mean value, $\langle\mathcal{A}\rangle$, is experimentally given, the estimated fidelity of the generated state to the desired state (Eq. (3)) takes the minimum of the fidelity values that are allowed by the given experimental data. For Bell-state estimation, our method exactly leads to the previous treatments that are intended to avoid the fake entanglement by Refs. [6] and [7]. Since our method does not rely on entanglement measure, which is not well established for more than 3-particle states, but relies on fidelity, our method is applicable not only to Bell states but also to n-particle states.

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[1] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academics, Dordrecht, The Netherlands 1989), pp. 69-72; see also N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[2] See, for example, M. Hillery *et al.*, Phys. Rev. A **59**, 1829 (1999).
[3] J. -W. Pan *et al.*, *Nature* **403**, 515 (2000); C. A. Sackett *et al.*, *Nature* **404**, 256 (2000); J. -W. Pan *et al.*, Phys. Rev. Lett. **86**, 4435 (2001)
[4] R. A. Fisher, Philos. Trans. R. Soc. London, Ser. A., **222**, 309 (1922).
[5] K. Banaszek *et al.*, Phys. Rev. A **61**, 010304(R) (1999).
[6] R. Horodecki *et al.*, Phys. Rev. A **59**, 1799 (1999).
[7] A. K. Rajagopal, Phys. Rev. A **60**, 4338 (1999).