On quantum computation with some restricted amplitudes

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Bernstein-Vazirani introduced the complexity classes **BQP** and **EQP** based on quantum Turing machines (QTMs), which are respectively the sets of languages recognized by bounded-error and exact polynomial-time QTMs. Afterward, they restricted their classes to the classes defined by QTMs with the amplitudes from the polynomial-time computable complex numbers. Adleman-DeMarrais-Huang introduced more general notions $\mathbf{BQP}_K$, $\mathbf{EQP}_K$ and $\mathbf{NQP}_K$, which are respectively the sets of languages recognized by bounded-error, exact and 'non-deterministic' polynomial-time QTMs with the amplitudes from a subset K of the complex number field $\mathbf{C}$. In recent years, complexity classes of QTMs with variously restricted amplitudes have been investigated. In particular, QTMs with amplitudes from the sets $\mathcal{A} = \{0, \pm 3/5, \pm 4/5, \pm 1\}$ and $\mathcal{B} = \{0, \pm 1/\sqrt{2}, \pm 1\}$ have been often used for the investigation of quantum complexity classes or quantum algorithms. Adleman-DeMarrais-Huang showed that although $\mathbf{BQP}_{\mathbf{C}}$ is uncountable, $\mathbf{BQP} = \mathbf{BQP}_{\mathcal{A}}$. Shor claimed that the following set of three gates, rotations around the x-axis and the z-axis by $\pi/2$, and Toffoli gates, are universal for quantum computation. This implies that $\mathbf{BQP} = \mathbf{BQP}_{\mathcal{B}}$ by the equivalence between bounded-error uniform quantum circuits and QTMs. On the other hand, Fenner-Green-Homer-Pruim showed $\mathbf{NQP}_{\overline{\mathbf{Q}}} = \mathbf{NQP}_{\mathcal{A}} = \mathbf{NQP}_{\mathcal{B}} =$ co-$\mathbf{C}_{=}\mathbf{P}$ and afterward Yamakami-Yao showed $\mathbf{NQP}_{\mathbf{C}} =$ co-$\mathbf{C}_{=}\mathbf{P}$. In this way, bounded-error and nondeterministic quantum complexity classes are not so sensitive to the choice of the amplitudes taken by QTMs. However, we can see that exact quantum complexity classes depend on the choice of the amplitudes. In this paper we explore exact and zero-error (or Las-Vegas) quantum complexity classes with some restricted amplitudes. Adleman-DeMarrais-Huang showed that for any $\theta \in [0, 2\pi)$ such that $\cos\theta$ is transcendental, $\mathbf{EQP}_{\{0, \pm\cos\theta, \pm\sin\theta, \pm 1\}} = \mathbf{P}$ and left the following question. *Does it hold* $\mathbf{EQP}_{\{0, \pm 3/5, \pm 4/5, \pm 1\}} = \mathbf{EQP}_{\{0, \pm 5/13, \pm 12/13, \pm 1\}}$*?* We give a general positive answer for this question; For any $\theta \in [0, 2\pi)$ such that $\cos\theta$ and $\sin\theta$ are rational, $\mathbf{EQP}_{\{0, \pm\cos\theta, \pm\sin\theta, \pm 1\}} = \mathbf{P}$. This result shows that the set $\mathcal{A}$ of amplitudes is useless for exact quantum computation, while it is sufficient for bounded-error quantum computation. Extending this result, we can show that if the bias of 'quantum coins' is rational, exact quantum computers with such biased coins have only equivalent power to deterministic computers. However, the set $\mathcal{B}$ seems to be so different from the set $\mathcal{A}$, since the quantum algorithm of Deutsch and Jozsa uses the amplitudes from $\mathcal{B}$. We also compare the utility of $\mathcal{A}$ with $\mathcal{B}$ from the viewpoint of zero-error algorithms. Combining Preskill's quantum circuit which produces the angle θ with $\cos\theta = 3/5$ probabilistically and the construction of universal QTMs, we show that QTMs with the amplitudes from $\mathcal{A}$ are not more powerful than ones with amplitudes from $\mathcal{B}$ for zero-error algorithms. On the other hand, we show that the factoring problem can be solved by a zero-error polynomial-time QTM with the amplitudes from $\mathcal{B}$.