A new proof of the direct part of quantum Stein's lemma

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1 Introduction

Let ρ and σ be density operators on a Hilbert space $\mathcal{H}$. We assume $d = \dim \mathcal{H} < \infty$ for mathematical simplicity. We study the hypothesis testing problem for the null hypothesis $\rho^{\otimes n} (=: \rho_n)$ versus the alternative hypothesis $\sigma^{\otimes n} (=: \sigma_n)$ on $\mathcal{H}^{\otimes n}$. In the sequel, an operator A_n on $\mathcal{H}^{\otimes n}$ satisfying inequalities $0 \le A_n \le I_n$ is called a test, since A_n is identified with the measurement $\{A_n, I_n - A_n\}$. For a test A_n, the error probabilities of the first kind and the second kind are, respectively, defined by $\alpha_n(A_n) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\,[\rho_n(I_n - A_n)]$ and $\beta_n(A_n) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\,[\sigma_n A_n]$. Let

$$\beta_n^*(\varepsilon) \stackrel{\mathrm{def}}{=} \min\{\beta_n(A_n) | 0 \le A_n \le I_n, \alpha_n(A_n) \le \varepsilon\}, \quad (1)$$

then the following quantum Stein's lemma holds.

$$\lim_{n\to\infty} \frac{1}{n} \log \beta_n^*(\varepsilon) = -D(\rho||\sigma), \quad (2)$$

where $D(\rho||\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Tr}\,[\rho(\log\rho - \log\sigma)]$ is the quantum relative entropy. First proof of (2) was composed of two parts. One part is the proof of the direct part:

$$\limsup_{n\to\infty} \frac{1}{n} \log \beta_n^*(\varepsilon) \le -D(\rho||\sigma) \quad (3)$$

by Hiai-Petz [1] and the other part is the proof of the converse part by Ogawa-Nagaoka [2]. A direct proof of the equality (2) was also given by Hayashi [3] using the information spectrum approach in quantum setting [4]. The purpose of this manuscript is to show a simple proof of (3). As is mentioned by Nagaoka [4], this proof also yields Hiai-Petz's theorem [1].

2 A new proof

We write the spectral decomposition of σ_n as $\sigma_n = \sum_j \lambda_{n,j} E_j^n$, where E_j^n is the projection to the eigenspace corresponding to an eigenvalue $\lambda_{n,j}$. Note that the number of eigenvalues, say v_n, is less than $(n+1)^d$. Define a map from the set of density operators to itself as $\mathcal{E}_n(\rho) \stackrel{\mathrm{def}}{=} \sum_j E_j^n \rho E_j^n$, then the map satisfies $\rho \le v_n \mathcal{E}_n(\rho)$ [3]. In the sequel, the state $\mathcal{E}_n(\rho_n)$ is simplified by $\overline{\rho_n}$. Since the function $x \mapsto -x^{-s} (0 \le s \le 1)$ is an operator monotone function, it holds that

$$\overline{\rho_n}^{-s} \le v_n^s \rho_n^{-s}. \quad (4)$$

Now, we focus on a test corresponding to the projection $\overline{S}_n(a) \stackrel{\mathrm{def}}{=} \{\overline{\rho_n} - e^{na}\sigma_n > 0\}$, where for a Hermitian operator $X = \sum_i x_i E_i$ the projection $\{X \ge 0\}$ is defined by $\{X \ge 0\} \stackrel{\mathrm{def}}{=} \sum_{i:x_i \ge 0} E_i$. The second error probability for the test $\overline{S}_n(a)$ is bounded as

$$\mathrm{Tr}\,[\sigma_n \overline{S}_n(a)] \le e^{-na} \mathrm{Tr}\,[\rho_n \overline{S}_n(a)] \le e^{-na}. \quad (5)$$

We can evaluate the first error probability as follows:

$$\begin{aligned} & \mathrm{Tr}\,[\rho_n(1 - \overline{S}_n(a))] \\ = \; & \mathrm{Tr}\,[\overline{\rho_n}(1 - \overline{S}_n(a))] \\ = \; & \mathrm{Tr}\,[\overline{\rho_n}^{1-s}\overline{\rho_n}^{s}(1 - \overline{S}_n(a))] \\ \le \; & e^{nas} \mathrm{Tr}\,[\overline{\rho_n}^{1-s}\sigma_n^s] \\ = \; & e^{nas} \mathrm{Tr}\,[\rho_n \sigma_n^{\frac{s}{2}} \overline{\rho_n}^{-s} \sigma_n^{\frac{s}{2}}] \\ \le \; & v_n^s e^{nas} \mathrm{Tr}\,[\rho_n \sigma_n^{\frac{s}{2}} \rho_n^{-s} \sigma_n^{\frac{s}{2}}] \\ = \; & \exp\left[n\left\{as - \psi(s) + \frac{s}{n}\log v_n\right\}\right], \end{aligned} \quad (6)$$

where the function $\psi(s)$ is defined as

$$\psi(s) \stackrel{\mathrm{def}}{=} -\log \mathrm{Tr}\,[\rho_n \sigma_n^{\frac{s}{2}} \rho_n^{-s} \sigma_n^{\frac{s}{2}}]. \quad (7)$$

Noting that $\psi(0) = 0$ and $\psi'(0) = D(\rho||\sigma)$, we can see that for all $a < D(\rho||\sigma)$ there exists $0 \le s \le 1$ such that $as - \psi(s) < 0$. Therefore, $\mathrm{Tr}\,[\rho_n(1 - \overline{S}_n(a))]$ goes to 0 with its exponent greater than $\max_{0\le s\le 1}\{-as + \psi(s)\} > 0$, since $\frac{s}{n}\log v_n \to 0\,(n \to \infty)$. The desired assertion follows from (5) and from the above fact.

References

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