Reversible extraction of classical information from a quantum source

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I. QUANTUM SOURCE CODING

A quantum source is described by an ensemble $\{\xi_i; p_i\}$ of (generally mixed) states ξ_i and probabilities p_i. Consider a source $\mathcal{E} = \{|\sigma_i\rangle\,; p_i\}$ of pure quantum signals, and blocks of n independent copies of the source:

$$\begin{aligned} |\sigma_I\rangle &= |\sigma_{i_1}\rangle \otimes \cdots \otimes |\sigma_{i_n}\rangle\,, \\ p_I &= p_{i_1} \cdots p_{i_n}\,, \end{aligned}$$

with $I = i_1 \ldots i_n$. Define $\rho = \sum_i p_i \sigma_i$ (we denote the projector $|\sigma_I\rangle\langle\sigma_I|$ simply as σ_I). The goal of source coding is that the combined encoding and decoding maps E and D (both completely positive, trace preserving, and linear), via a coding system consisting of k qubits, has fidelity

$$F(E,D) = \sum_I p_I \mathrm{Tr}\,(\sigma_I D(E(\sigma_I))) \geq 1 - \epsilon \longrightarrow 1,$$

as $n \to \infty$. The quantum source coding theorem [1, 2] says that the infimum of the rate $\alpha = k/n$ is the von Neumann entropy of ρ: $S = S(\rho) = -\mathrm{Tr}\,\rho \log \rho$.

II. EXTRACTING CLASSICAL INFORMATION

Suppose we still want to reduce the quantum storage resources (for example, because quantum memory is hard to stabilize). The idea is to store part of the information classically: classical memories are cheap and reliable. This classical information is extracted at the encoding stage, and — since it can be copied — it may be around still after decoding. Thus the process yields partial information about the ensemble state, in an asymptotically disturbance free fashion. This is what we call *reversible extraction of classical information.*

Formally the encoding map is such that the encoded states are written

$$E(\sigma_I) = \sum_j p(j|I)\,|j\rangle\langle j| \otimes \omega_j^I,$$

with generally mixed states ω_j^I on a k_j–qubit system.

For example if the states σ_i are orthogonal, the encoder may measure them and send the information classically, so no quantum memory is needed altogether. One might wonder if for an ensemble of two slightly nonorthogonal states there might be some gain in a similar way.

However, it holds:

Theorem: If the source is *irreducible* (the states do not fall into orthogonal classes) then, with $\overline{k} = \sum_j p_j k_j$, $\liminf \overline{k}/n \geq S$, for every sequence of coding schemes with limiting fidelity 1.

A predecessor of this theorem was in [3], where the case of zero error was solved.

A *proof* of this is outlined as follows: the Holevo information of the ensemble can never increase. It is $nS(\rho)$ initially, and close to $nS(\rho)$ after decoding. But the encoded ensembles' Holevo information can be expressed as the sum if $I(I \wedge j)$ and the average Holevo information of ensembles $\{\omega_j^I; p(I|j)\}$, which in turn is upper bounded by $\overline{k}$. Hence all we have to show is that $I(I \wedge j)/n \to 0$. Then we use an inequality familiar classically,

$$I(i_1 \ldots i_n \wedge j) \leq \sum_t I(i_t \wedge j | I_{\neq t}).$$

To bound each of the terms we use a specially designed information–disturbance trade–off showing that each term tends to zero as $\epsilon \to 0$.

For a *reducible* source (i.e., states falling into two or more orthogonal subspaces) we can similarly show that the maximum classical information reversibly extractable is obtained by the von Neumann measurement corresponding to these subspaces.

III. AN ALGEBRAIC APPROACH

Since the completion of this work, papers by Koashi and Imoto have appeared [4], which lead to the same conclusion as ours from a quite different angle (and, at the same time, to an extension to mixed state sources). Their result can be obtained by exploiting an interesting connection to the theory of representations of von Neumann algebras on Hilbert spaces, based on a lucid exposition by Lindblad [5]:

We use the same reduction to single letters as before, but observe that in the asymptotic limit the combined en– and decoding converge to a map that is the identity on the source states. Then we use algebraic methods to characterize such maps (in the irreducible source case there is only one, the identity), and argue by continuity.

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[3] C. H. Bennett, G. Brassard, R. Jozsa, D. Mayers, A. Peres, B. Schumacher and W. Wootters, "Reduction of quantum entropy by reversible extraction of classical information", *J. Mod. Optics*, **41**, pp. 2307–2314 (1994).

[4] M. Koashi, N. Imoto, "What is Possible Without Disturbing Quantum Signals?", quant-ph/0101144, and "Teleportation cost and hybrid compression of quantum signals", quant-ph/0104001.

[5] G. Lindblad, "A general no-cloning theorem", *Lett. Math. Phys.*, **47**, no. 2, pp. 189–196, 1999.