Compressibility and nature of information

Masato Koashi
CREST Research Team for Interacting Carrier Electronics, School of Advanced Sciences, The Graduate University for Advanced Studies (SOKENDAI), Hayama, Kanagawa, 240-0193, Japan

One of the fundamental questions in the information theory is about the data compression, namely, what is the shortest description of the data. One obvious goal of this approach is the quantification of the amount of information, and for this goal we will seek the shortest description length among various compression scenarios. In addition to this, there is another motivation — understanding of how well we can manipulate information stored in physical systems, which is a central topic in the field of quantum information. For this aim, we are interested in how the optimal compression rates differ in various compression scenarios in which the conditions posed to the compressor are slightly different. We here report that the optimal rates of three different scenarios of compression clearly reflect the nature (classical/quantum; pure/mixed) of information to be compressed.

A powerful tool that is useful to determine the optimal rates in blind scenarios is the theory [1] of the operations that preserve a given set of initial states $\{\hat{\rho}_i\}$. Given $\{\hat{\rho}_i\}$, we can find a unique decomposition of $\mathcal{H}_\mathrm{A}$, the support of $\hat{\rho}$, written as $\mathcal{H}_\mathrm{A} = \bigoplus_l \mathcal{H}_\mathrm{J}^{(l)} \otimes \mathcal{H}_\mathrm{K}^{(l)}$. Under this decomposition, $\hat{\rho}_i$ is written as $\hat{\rho}_i = \bigoplus_l p^{(i,l)} \hat{\rho}_\mathrm{J}^{(i,l)} \otimes \hat{\rho}_\mathrm{K}^{(l)}$, where $\hat{\rho}_\mathrm{J}^{(i,l)}$ and $\hat{\rho}_\mathrm{K}^{(l)}$ are normalized density operators acting on $\mathcal{H}_\mathrm{J}^{(l)}$ and $\mathcal{H}_\mathrm{K}^{(l)}$, respectively, and $p^{(i,l)}$ is the probability for the state to be in the subspace $\mathcal{H}_\mathrm{J}^{(l)} \otimes \mathcal{H}_\mathrm{K}^{(l)}$. Note that $\hat{\rho}_\mathrm{K}^{(l)}$ is independent of i. Any operation $\hat{U}$ on $\mathcal{H}_\mathrm{A}$ and the ancilla system $\mathcal{H}_\mathrm{E}$ that preserves $\{\hat{\rho}_i\}$ is then expressed in the form $\hat{U} = \bigoplus_l \hat{1}_\mathrm{J}^{(l)} \otimes \hat{U}_\mathrm{KE}^{(l)}$, where $\hat{U}_\mathrm{KE}^{(l)}$ are unitary operators acting on $\mathcal{H}_\mathrm{K}^{(l)} \otimes \mathcal{H}_\mathrm{E}$.

This theorem is applied to the problems of compressing a source $\mathcal{E} = \{p_i, \hat{\rho}_i\}$ in the following way. The decomposition $\hat{\rho}_i = \bigoplus_l p^{(i,l)} \hat{\rho}_\mathrm{J}^{(i,l)} \otimes \hat{\rho}_\mathrm{K}^{(l)}$, together with the decomposition of the averaged state $\hat{\rho} \equiv \sum_i p_i \hat{\rho}_i = \bigoplus_l p^{(l)} \hat{\rho}_\mathrm{J}^{(l)} \otimes \hat{\rho}_\mathrm{K}^{(l)}$, gives a particular scheme to compress it, and hence tells us an upper bound of the optimal rate. On the other hand, the fact that any faithful compression-decompression scheme must be written as $\hat{U} = \bigoplus_l \hat{1}_\mathrm{J}^{(l)} \otimes \hat{U}_\mathrm{KE}^{(l)}$ tells us that we can send a certain amount of classical information through $\mathcal{H}_\mathrm{J}^{(l)}$, which gives a lower bound of the compression rate.

In the case of the blind variable-length and faithful (VLF) scenario, the above strategy gives [2] the optimal regularized rate $R_\mathrm{opt}^{(\infty)}(\mathcal{E})$ as

$$R_\mathrm{opt}^{(\infty)}(\mathcal{E}) = I_\mathrm{C}(\mathcal{E}) + D_\mathrm{NC}(\mathcal{E}), \tag{1}$$

where $D_\mathrm{NC}(\mathcal{E}) \equiv \sum_l p^{(l)} \log_2 \dim \mathcal{H}_\mathrm{J}^{(l)}$ and $I_\mathrm{C}(\mathcal{E}) \equiv -\sum_l p^{(l)} \log_2 p^{(l)}$. For the classical part associated with the index l, the rate is given by its Shannon entropy. On the other hand, the internal state of each $\mathcal{H}_\mathrm{J}^{(l)}$, which is regarded as the genuinely quantum part of the ensemble $\mathcal{E}$, is incompressible in this absolutely faithful scenario.

In the case of the blind fixed-length and asymptotically faithful (FLAF) scenario, the optimal rate $R_\mathrm{opt}^\mathrm{AF}(\mathcal{E})$ is

$$R_\mathrm{opt}^\mathrm{AF}(\mathcal{E}) = I_\mathrm{C}(\mathcal{E}) + I_\mathrm{NC}(\mathcal{E}), \tag{2}$$

where $I_\mathrm{NC}(\mathcal{E}) \equiv \sum_l p^{(l)} S(\rho_\mathrm{J}^{(l)})$ [3]. This rate is equal to the von Neumann entropy of the ensemble made up of the reduced states $\hat{\rho}_i^\mathrm{R} \equiv \bigoplus_l p^{(i,l)} \hat{\rho}_\mathrm{J}^{(i,l)}$, obtained from $\hat{\rho}_i$ by removing the redundant part $\hat{\rho}_\mathrm{K}^{(l)}$.

The difference $R_\mathrm{opt}^{(\infty)} - R_\mathrm{opt}^\mathrm{AF}$ is generally nonzero. For the classical ensembles in which all $\hat{\rho}_i$ commutes, this gap vanishes. This is in good contrast with the gap between $R_\mathrm{opt}^\mathrm{AF}$ and I_eff, the optimal rate of the visible scenario in which the classical index i is given to the compressor, rather than the state $\hat{\rho}_i$ itself. This gap is upper-bounded by $\sum_i p_i S(\hat{\rho}_i^\mathrm{R})$, and vanishes when the letter states are effectively pure. The gap can be nonzero even if all $\hat{\rho}_i$ commute. These observations are summarized in Table I. When the letter states are distinct, namely, $\{\hat{\rho}_i\}$ are all orthogonal, the optimal compression rates for the three scenarios are equal. When the letter states overlap with each other and are not completely distinguishable, there can be two types of overlaps. One is the cases when each letter state is noisy. In such cases, whether or not the identity of each letter is given as side information affect the compressibility. The other way of overlapping is quantum one, where the letter states are pure and suffers no noises, but are not completely distinguishable from each other due to nonorthogonality. In such cases, the compressibility depends on whether small errors (even an asymptotically vanishing one) are allowed or not.

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Information	classical ($[\hat{\rho}_i, \hat{\rho}_j] = 0$)	quantum
pure ($S(\hat{\rho}_i^\mathrm{R}) = 0$)	$R_\mathrm{opt}^{(\infty)} = R_\mathrm{opt}^\mathrm{AF} = I_\mathrm{eff}$	$R_\mathrm{opt}^{(\infty)} \geq R_\mathrm{opt}^\mathrm{AF} = I_\mathrm{eff}$
mixed	$R_\mathrm{opt}^{(\infty)} = R_\mathrm{opt}^\mathrm{AF} \geq I_\mathrm{eff}$	$R_\mathrm{opt}^{(\infty)} \geq R_\mathrm{opt}^\mathrm{AF} \geq I_\mathrm{eff}$

TABLE I. Nature of information and gaps between optimal compression rates for various scenarios —the blind VLF ($R_\mathrm{opt}^{(\infty)}$), the blind FLAF ($R_\mathrm{opt}^\mathrm{AF}$), and the visible FLAF (I_eff).

[1] to appear in Phys. Rev. A, quant-ph/0101144v2.
[2] Phys. Rev. Lett, **89**, 097904 (2002).
[3] Phys. Rev. Lett. **87**, 017902 (2001).