Classification of multipartite entanglement by the hyperdeterminant

Akimasa Miyake and Miki Wadati

Department of Physics, Graduate School of Science, University of Tokyo,
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

Tel/Fax: +81-3-5841-4150

Email: miyake@monet.phys.s.u-tokyo.ac.jp, wadati@monet.phys.s.u-tokyo.ac.jp

Entanglement is the quantum correlation exhibiting nonlocal (nonseparable) properties. It is supposed to be never strengthened, on average, by local operations and classical communication (LOCC). In particular, entanglement in *multi*-parties is of fundamental interest in quantum many-body theory, and makes quantum information processing (QIP), e.g., distillation protocol, more efficient than that relying on entanglement only in *two*-parties. Here, we classify and characterize the multipartite entanglement which has yet to be understood, compared with the bipartite one.

When we classify the single copy of multipartite pure states on the Hilbert space $\mathcal{H} = \mathbb{C}^{k_1+1} \otimes \cdots \otimes \mathbb{C}^{k_n+1}$ (precisely, its projective space $M = \mathbb{C}P^{(k_1+1)\cdots(k_n+1)-1}$),

$$|\Psi\rangle = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle,$$

there are many difficulties in applying the techniques, e.g., the Schmidt decomposition, utilized in the bipartite case. Still, we can consider a coarser classification by stochastic LOCC (SLOCC) than LOCC. There we identify two states $|\Psi\rangle$ and $|\Phi\rangle$ that convert to each other back and forth with (maybe different) nonvanishing probabilities, in contrast with LOCC where we identify the states interconvertible deterministically. These states $|\Psi\rangle$ and $|\Phi\rangle$ are supposed to perform the same tasks in QIP although their probabilities differ. Then the SLOCC classification is equivalent to the classification of orbits of the action: direct product of general linear groups $GL_{k_1+1}(\mathbb{C})\times\cdots\times GL_{k_n+1}(\mathbb{C})$. In the bipartite case, the SLOCC classification means just classifying the whole states M by the Schmidt local rank, i.e., the rank of a_{i_1,i_2}. In the 3-qubit case, Dür *et al.* showed that SLOCC classifies M into finite classes and in particular there exist two inequivalent, Greenberger-Horne-Zeilinger (GHZ) and W, classes of the genuine tripartite entanglement. They also pointed out that the SLOCC classification has infinitely many orbits in general (e.g., for $n \geq 4$).

In this talk, we classify multipartite entanglement in a unified manner based on the hyperdeterminant. The advantages are three-fold.

1. This classification is equivalent to the SLOCC classification when SLOCC has finitely many orbits. So it naturally includes the widely known bipartite and 3-qubit cases.

2. In the multipartite case, we need further SLOCC invariants in addition to the local ranks. For example in the 3-qubit, the 3-tangle τ, just the absolute value of the hyperdeterminant, is utilized to distinguish GHZ and W classes. This work clarifies why τ appears and how these SLOCC invariants are generally related to the hyperdeterminant.

3. It is also useful to the classification of multipartite mixed states (for details, see the Appendix A of the reference). We concentrate the pure states here.

Our idea is based on a duality between the set of separable states and the set of entangled states. Precisely, as seen in the left figure below, the set of completely separable states is the smallest (innermost) closed subvariety, called Segre variety, X. Its dual variety $X^\vee$ is the largest (outermost) closed subvariety which consists of *degenerate* entangled classes. In the bipartite ($k_1 = k_2$) case, it means that X is the set of states of the Schmidt local rank 1, and $X^\vee$ is the set of states where the Schmidt local rank is not full, i.e., $\det A = 0$. In general, the dual variety $X^\vee$, if 1-codimensional, is defined by the zero hyperdeterminant: $\mathrm{Det}A = 0$. The outside of $X^\vee$, given by $\mathrm{Det}A \neq 0$, is the generic (non degenerate) entangled class, and $|\mathrm{Det}A|$ is the entanglement measure which represents the amount of generic entanglement. It is also known as the concurrence C, 3-tangle τ for the $2, 3$-qubit pure case, respectively.

$$
\begin{aligned}
C &= 2|\mathrm{Det}A_2| = 2|\det A| = 2|a_{00}a_{11} - a_{01}a_{10}|, \\
\tau &= 4|\mathrm{Det}A_3| = 4|a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2(a_{000}a_{001}a_{110}a_{111} \\
&\quad + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{100}a_{011}a_{110} \\
&\quad + a_{010}a_{100}a_{011}a_{101}) + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})|.
\end{aligned}
$$

It is significant that $\mathrm{Det}A$ is relatively invariant under SLOCC. In order to classify the degenerate classes in $X^\vee$, we need to calculate the singularities of $X^\vee$.

This is how we obtain the onion picture of entangled classes under SLOCC. The noninvertible local operations, where the ranks of the action for certain parties are not full, degrade the entangled classes, and give the partially ordered structure of multipartite entanglement (cf. the right figure). We will clarify what this structure looks like, when the dimensions k_j+1 of subsystems become larger, or when the number n of the parties increases. Moreover, we will reveal a salient difference from the bipartite case. In the bipartite and exceptionally 3-qubit cases, the maximally entangled states in Bell's inequalities are the generic class of $\mathrm{Det}A \neq 0$. However, it is not the case in general, e.g., in the $n \geq 4$-qubit. It implies that the majority of multipartite entangled states never locally converts to the maximally entangled states in Bell's inequalities (GHZ) even probabilistically.

Reference: A. Miyake, quant-ph/0206111.

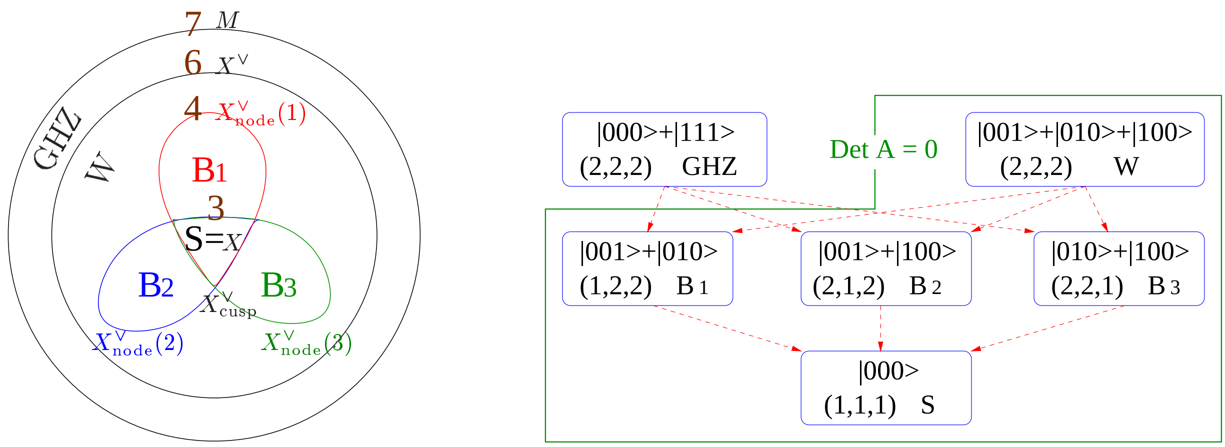

FIG. 1: [Left] The onion-like classification of SLOCC orbits (entangled classes) in the 3-qubit case. There is the duality between the innermost closed subvariety X and the outermost closed subvariety $X^\vee$. [Right] The partially ordered structure of 3-qubit pure entangled states under noninvertible local operations (indicated by dashed arrows). Each class, corresponding to the SLOCC orbit, is labeled by the representative, local ranks (equivalently, singularities of $X^\vee$), and the name.