CPTP Mappings, Repeaters in Lossy Quantum Channels and State-dependent Quantum Cloning

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I. ABSTRACT

Suppose that we receive a quantum state which is drawn from a parametrized set $\{\hat{f}_i\}$ with known a priori distribution, $\{p_i\}$, and that we have another set with known a priori distribution, $\{\hat{f}_i\}$ and that we have another set of states, which we call *templates*, $\{\hat{g}_i\}$ at our disposal. Our task is to output an appropriate state of a classical mixture of the $\hat{g}_i$'s so as to maximize a proper score function. Namely, we are to design the best quantum process, generally a CPTP map acting on the input $\hat{f}_i$, that outputs a quantum state $\hat{\rho}_i$ of the form

$$CPTP : \hat{f}_i \mapsto \hat{\rho}_i = \sum_j p_{ij}\hat{g}_j, \tag{1}$$

such that a suitable average score, e.g.

$$\bar{S} = \sum_i p_i F(\hat{f}_i, \hat{\rho}_i), \tag{2}$$

is maximized. Here $F(\hat{\rho}, \hat{\sigma}) \equiv [\,\mathrm{Tr}\,[\hat{\rho}^{1/2}\hat{\sigma}\hat{\rho}^{1/2}]^{1/2}]^2$ is the fidelity between two mixed states $\hat{\rho}$ and $\hat{\sigma}$. This is a basic task in quantum information processing. For example, this can be the mathematical modelling for an eavesdropping strategy in a quantum cryptosystem, an action of a quantum repeater in a communication channel, a state-dependent cloning process, and so on. In this letter, the existence conditions for CPTP maps between two binary sets of quantum state is presented in a geometrical picture. By using these results, we derive the optimal strategy for the CPTP mapping that maximizes the score given by Eq. (2) in a simplified model in which the input states are chosen to belong to the one parameter family of nonorthogonal and normalized binary pure states $|f_i\rangle \equiv \alpha_i|0\rangle + \beta_i|1\rangle$, the *templates* are the binary mixed states

$$\hat{g}_i \equiv \frac{1}{2}[\hat{I} + \vec{g_i}\cdot\hat{\sigma}] \tag{3}$$

(where we take, without lack of generality, $\vec{g_1}^2 = \vec{g_2}^2 = g^2$, and $g \in [0,1]$), and the output states are given as a classical mixture of the $\hat{g}_i$'s as follows

$$\begin{aligned}\hat{\rho}_1 &= p\,|g_1\rangle\langle g_1| + (1-p)\,|g_2\rangle\langle g_2|,\\ \hat{\rho}_2 &= q\,|g_2\rangle\langle g_2| + (1-q)\,|g_1\rangle\langle g_1|,\end{aligned} \tag{4}$$

with the output probability distributions $(p,q) \in [0,1]$.

The necessary and sufficient condition for the existence of a CPTP linear mapping between a couple of 2-d density operators representing $\{|f_1\rangle, |f_2\rangle\}$ and another couple of 2-d density operators representing $\{\hat{\rho}_1, \hat{\rho}_2\}$ is the well known fidelity criterion [1]

$$F(\hat{f}_1, \hat{f}_2) \leq F(\hat{\rho}_1, \hat{\rho}_2). \tag{5}$$

Given the output states (4), it is easy to evaluate the fidelities so that the CPTP mapping existence condition (5) can be explicitly rewritten as

$$\begin{aligned}\sqrt{Y_0-(2p-1)^2}\sqrt{Y_0-(2q-1)^2} &\geq (2p-1)(2q-1)\\ &+ Y_0 - 2R,\end{aligned} \tag{6}$$

where the parameters

$$R \equiv \frac{f^2\sin^2\phi}{g^2\sin^2\theta} \geq 0\ ;\ Y_0 \equiv 1 + \frac{1-g^2}{g^2\sin^2\theta} \geq 1, \tag{7}$$

with $2\sin^2\theta \equiv 1 - \vec{g_1}\cdot\vec{g_2}/g^2$, $2\sin^2\phi \equiv 1 - \vec{f_1}\cdot\vec{f_2}/f^2$ and $\phi, \theta \in [0,\pi]$.

After some straightforward algebra one finally obtains that the Uhlmann fidelity criterion can be satisfied in certain geometrically simple (ellipse curves) (p,q) parameter regions, classified according to the values of R and Y_0.

The problem of maximizing the average score (2) under the condition of the existence of the CPTP map can then be easily solved by use of the standard Lagrange multiplier method, obtaining the optimal pair for pure templates $\{|g_i\rangle\}$ (i.e., $Y_0 = 1$)

$$p_{opt} = \frac{1}{2}\left[1 + \frac{c_-}{\sqrt{c}}\right]\ ;\ q_{opt} = \frac{1}{2}\left[1 + \frac{c+-}{\sqrt{c}}\right], \tag{8}$$

for $0 < R < 1$, where $c_\pm \equiv R \pm (R-1)(1-2\xi)$ and $c \equiv (1-2\xi)^2 + 4\xi(1-\xi)R$.

An interesting problem that can be naturally addressed within the context of our CPTP analysis is the following. Consider a binary set of coherent states $\{|\alpha_0\rangle, |-\alpha_0\rangle\}$ as our input signals $\{|f_1\rangle, |f_2\rangle\}$, each arising with a priori probabilities $1-\xi$ and ξ, respectively, which are sent by Alice through a lossy quantum channel $\mathcal{L}$. We are at an intermediate station (Eve) in a quantum communications channel, and have at our disposal another set

of coherent states $\{|\beta\rangle, |-\beta\rangle\}$, which correspond to our template states $\{|g_1\rangle, |g_2\rangle\}$, and this set is more orthogonal than the input, i.e. $|\beta| > |\alpha|$, where $|\pm\alpha\rangle \equiv |\pm\eta\alpha_0\rangle$. Now, we would like to know the communication performance through the lossy channel by direct transmission of $\{|\alpha\rangle, |-\alpha\rangle\}$ (no action by Eve)

$$\begin{aligned} \hat{\mathcal{L}}(|\alpha\rangle\langle\alpha|) &\mapsto \hat{\rho}_{\alpha,1} \equiv |\eta\alpha\rangle\langle\eta\alpha|, \\ \hat{\mathcal{L}}(|-\alpha\rangle\langle-\alpha|) &\mapsto \hat{\rho}_{\alpha,2} \equiv |-\eta\alpha\rangle\langle-\eta\alpha|, \end{aligned} \tag{9}$$

where $0 < \eta < 1$, and compare it with the performance in the case when Eve performs an optimal CPTP amplification before sending the states to Bob, i.e. when

$$\begin{aligned} \hat{\mathcal{L}}(\hat{\rho}_1) &\mapsto \hat{\rho}'_1 \equiv p\,|\eta\beta\rangle\langle\eta\beta| + (1-p)\,|-\eta\beta\rangle\langle-\eta\beta|, \\ \hat{\mathcal{L}}(\hat{\rho}_2) &\mapsto \hat{\rho}'_2 \equiv q\,|-\eta\beta\rangle\langle-\eta\beta| + (1-q)\,|\eta\beta\rangle\langle\eta\beta| \end{aligned} \tag{10}$$

where

$$\begin{aligned} \hat{\rho}_1 &\equiv p\,|\beta\rangle\langle\beta| + (1-p)\,|-\beta\rangle\langle-\beta|, \\ \hat{\rho}_2 &\equiv q\,|-\beta\rangle\langle-\beta| + (1-q)\,|\beta\rangle\langle\beta|, \end{aligned} \tag{11}$$

and the parameters p and q are evaluated in Eq. (11) by Eve as the best pair given by Eq. (8). The communication performance can be evaluated by comparing the minimum average error probability P_e (the so called Helstrom bound) for the cases when Bob receives the states (9) or (10) and he performs an optimal POVM $\{\hat{\Pi}_i\}$ $(i = 1, 2)$ trying to discern whether the state $|\alpha\rangle$ or $|-\alpha\rangle$ was originally sent by Alice. It can be simply proved that the optimal error probability $P_{e,CPTP}$ is always smaller than $P_{e,triv}$ for any choice of initial probability distributions ξ, $0 < \eta < 1$ and $|\beta| > |\alpha|$. That is, the intermediate action by Eve with optimal CPTP mapping on the initial states reduces the final error probability of detecting the original states by Bob. We also consider the performance of the channel with respect to the Holevo capacities $I_{H,CPTP}$ and $I_{H,triv}$, and show that there exists both parameters $(\beta/\alpha, \eta)$ region where $I_{H,CPTP} > I_{H,triv}$ and $I_{H,CPTP} < I_{H,triv}$. A possible interpretation for the different behavior between P_e and I_H may be given since the Helstrom bound specifies the performance of separate measurements, while the Holevo capacity is a measure for the coding by large scale collective measurements. So preparing mixed signals at the repeater could spoil in some cases the potential of quantum collective decoding that is involed in the pure state signals $\{|\eta\alpha\rangle, |-\eta\alpha\rangle\}$.

Another interesting application of the CPTP mapping results is in state-dependent cloning. In particular, we assume that the input states are pure and given as an N-fold tensor product $|f_i\rangle^{\otimes N}$, while the 'templates' $\hat{g}_i$ given by Eq. (3) are pure $(g = 1)$ and M-copies clones $(M \geq N \geq 1)$ of the input states $|f_i\rangle$, i.e.

$$|f_i\rangle \to \left|\tilde{f}_i\right\rangle \equiv |f_i\rangle^{\otimes N} \quad ; \quad |g_i\rangle \equiv |f_i\rangle^{\otimes M}. \tag{12}$$

We then restrict our analysis to the special case in which we assumed that we are only able to construct outputs which are classical mixtures of these templates, that is the outputs are given again by Eq. (4). In order to evaluate the efficiency of the cloning machine, we can now either choose as the figure of merit the 'global' fidelity

$$\bar{F}_G \equiv (1-\xi)\left\langle\tilde{f}_1|\hat{\rho}_1|\tilde{f}_1\right\rangle + \xi\left\langle\tilde{f}_2|\hat{\rho}_2|\tilde{f}_2\right\rangle, \tag{13}$$

(essentially the same as the score (2)), otherwise, we could choose the 'local' fidelity

$$\bar{F}_L \equiv (1-\xi)F_1(\hat{f}_1, \hat{f}_1^{out}) + \xi F_2(\hat{f}_2, \hat{f}_2^{out}), \tag{14}$$

where $\hat{f}_i^{out}$ is the reduced density operator for one single copy of the final state. A short calculation shows that the 'local' and 'global' fidelities are linearly correlated and have the same optimal solution. Another measure of the quality of the performance of our copier can be given in terms of the Holevo bound on the copied information for the reduced density outputs. The evident advantage of our optimal CPTP mapping method in a general cloning machine relies in not having to deal with all the inequalities which derive from the constraints on the unitarity of transformations over extended Hilbert spaces with ancilla qubits, as we just have to maximize the chosen figure of merit along a certain curve specifying the boundary of the allowed CPTP mappings between the initial and the output (mixed) states.

Completely similar considerations can be extended to the case in which the input and the template states are, respectively, the coherent states $|\pm\alpha\rangle$ and $|\pm\beta\rangle$, or to the case of a special type of copier called $N \to K + L$ (with $K + L \geq N$) 'anti-cloning' machine.

[1] A. Uhlmann, Per. Math. Phys. 9, 273 (1976).