Quantum Cellular Automata on Infinite Configurations

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Motivation. Cellular automata (CA) are a parallel model of universal computation popular in modelling complex systems. They consist of a regular array of individually simple computing elements (the cells) and compute through uniform, local, and synchronous interaction. Quantum cellular automata (QCA) are the quantum analogue of classical CA. They can be used to describe and model quantum systems, such as quantum lattice gases. In particular, some of the architectures for quantum computers (e.g., NMR) resemble CA. This (aside from purely theoretical interest) is why several models of QCA have been suggested. My approach, using observables to represent the states of cells, also underscores the connection with quantum spin systems.

Previous work. Several models of QCA have been proposed. Some of them choose to model only the movements of a finite number of particles on a lattice, while the rest take a more general approach: Let Q be a finite set of basis states and $N \in \mathbb{N}$ the neighbourhood size. Without loss of generality, we will assume N to be odd. The local update function φ is a mapping $Q^N \to \ell_2(Q)$. Q is assumed to contain a *quiescent* state q_s with $\varphi(q_s^N) = q_s$. A global configuration is an element c of $Q^{\mathbb{Z}}$; we write c_i for the state of the cell with index i. Let $Q_E^{\mathbb{Z}}$ be the set of finite global configurations, i.e., those where all but a finite number of cells are in state q_s. A finite quantum global configuration is a superposition $\sum_{i=1}^n \alpha_i |c^{(i)}\rangle$ with $\alpha_i \in \mathbb{C}$, $c^{(i)} \in Q_E^{\mathbb{Z}}$ and $\sum_{i=1}^n |\alpha_i|^2 = 1$.

The local update function induces a global update function $\Phi : Q_E^{\mathbb{Z}} \to \ell_2(Q)_E^{\mathbb{Z}}$ by $\Phi(c)_i = \varphi(c_{i-(N-1)/2}, \cdots, c_{i+(N-1)/2})$; for $c \in Q_E^{\mathbb{Z}}$, $\Phi(c)$ is the finite quantum global configuration $\sum_{i=1}^{n} \alpha_i |c^{(i)}\rangle$ where $\alpha_i = \prod_{j\in\mathbb{Z}} \langle \Phi(c)_j | c_j^{(i)} \rangle$. Since c is a finite configuration, each α_i contains only a finite number of factors different from 1. Φ can be continued linearly to work on finite quantum global configurations. The local update functions of QCA are required to induce unitary global update functions. For the models by Watrous, van Dam and Meyer, there are criteria on the local update function that are equivalent with global unitarity.

QCA and $C^\star$-algebras. Instead of rays in a Hilbert space, I write the states of cells, blocks of cells, and global configurations as observables. This has the advantage of introducing a useful notion of local state (which is not present in the other definitions of QCA). The set of observables of a quantum system forms a $C^\star$-algebra. $\star$-automorphisms are structure-preserving bijective endomorphisms of $C^\star$-algebras. Unitary operators U induce a special case of $\star$-automorphisms called inner automorphisms by $a \mapsto UaU^\dagger$. Let $\mathcal{O}(\ell_2(Q)^k)$ be the observable algebra on blocks of k cells. There is a canonical embedding of $\mathcal{O}(\ell_2(Q)^k)$ into $\mathcal{O}(\ell_2(Q)^{k+1})$ for all $k \in \mathbb{N}$. The sequence $\mathcal{O}(\ell_2(Q)), \mathcal{O}(\ell_2(Q)^2), \mathcal{O}(\ell_2(Q)^3), \ldots$ has a limit, whose completion (in the metric induced by the operator norm) is a $C^\star$-algebra called the quasilocal algebra $\mathcal{A}_{ql}$.

I use elements of the quasilocal algebra to represent the states of infinite configurations. This makes it possible to define block quantum CA (BQCA) using a finite set Q of cell states, a block length N, a sequence of origins $i_1, i_2, \ldots, i_k$, and a $\star$-automorphism φ of $\mathcal{O}(\ell_2(Q)^N)$. BQCA define $\star$-automorphisms of the quasilocal algebra. Moreover, they can simulate the partitioned QCA by Watrous and the QCA by van Dam and vice versa. Thus, the advantage of this approach does not lie in the power of BQCA, but in their ability to represent local states including the limited amount of knowledge that a cell may have of its own state and that of its neighbours. This not only leads to an extension to infinite configurations, but it also makes it possible to define useful metrics on global configurations.

I suggest one such metric that is quite close to the Cantor metric usually applied to classical CA and use it to prove a characterisation result paralleling a well-known theorem by Hedlund: QCA implement precisely the continuous, shift-commuting automorphisms of the space of infinite global configurations. This implies that whenever a quantum computable phenomenon arises from uniform and local interaction, it can be realised by a QCA.