Remarks on additivity of the Holevo channel capacity and of the entanglement of formation

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1 Introduction

Holevo capacity of a quantum channel and *entanglement of formation* [1] of a quantum state raise the natural problem of *additivity* under tensor products.

The literature on the subject is vast and increasing fastly, and in the presentation, we will point out that the Stinespring dilation of a completely positive map provides the link between the two quantities, which will be exploited in a number of examples, some involving group symmetry arguments. Some of these results are used to demonstrate a gap between entanglement cost and distillable entanglement.

We also discuss the relation of *superadditivity* of entanglement of formation, most notably its implying additivity of entanglement of formation, of channel capacity, and of channel capacity with a linear cost constraint.

2 Holevo capacity $C(T)$ and entanglement of formation $E_f(\rho)$

Holevo capacity, or the classical capacity $C(T)$ of a quantum channel$T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}_2)$, with $\mathcal{H}$ and $\mathcal{H}_2$ being Hilbert spaces, is given by

$$C(T) = \sup_{\{p_i,\pi_i\}} I\big(p; T(\pi)\big)$$

where $\{p_i, \pi_i\}$ runs over all the pure state ensemble on $\mathcal{H}$, $I(p; \rho)$ is Holevo mutual information, and $S(\omega)$ is von Neumann entropy[6]. The *entanglement of formation* $E_f(\rho)$ [1] of a state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as

$$E_f(\rho) := \inf_{\{p_i,\rho_i\}} \sum_i p_i S\left(\mathrm{Tr}_{\mathcal{H}_2}\pi\right),$$

where inf is taken over all the pure state ensembles $\{p_i, \rho_i\}$ with $\sum_i p_i\rho_i = \rho$.

It is conjectured that both of these quantities are additive (see [5] and the above references),

$$C(T_1 \otimes T_2) = C(T_1) + C(T_2), \quad (1)$$
$$E_f(\rho_1 \otimes \rho_2) = E_f(\rho_1) + E_f(\rho_2). \quad (2)$$

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While (1) proved for the cases like,

(I) unital qubit–channels [7, 8],
(II) arbitrary depolarising channels [2, 3, 9],
(III) entanglement–breaking channels [12],

(2) is proved only in a few cases the only published examples are in [14]. In our presentation, (1) is related to (2), producing several new examples in which (2) is valid.

If the additivity of entanglement of formation would turn out to be true, the *entanglement cost* $E_c(\rho)$ of ρ, i.e. the asymptotic rate of EPR pairs to approximately create n copies of ρ is given by $E_f(\rho)$, for we have [4],

$$E_c(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f\big(\rho^{\otimes n}\big).$$

3 Stinespring dilation: Linking $C(T)$ to $E_f(\rho)$

Due to a theorem of Stinespring [13] the TPCP map T can be presented as the composition of an isometric embedding of $\mathcal{H}$ into a bipartite system with a partial trace. By embedding into larger spaces we can present U as restriction of a unitary, which often we silently assume done. Denote $\mathcal{K} := U\mathcal{H} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$, the image subspace of U. Then we can say that T is equivalent to the partial trace channel, with inputs restricted to states on $\mathcal{K}$. This entails:

$$C(T) = \sup\{S\big(\mathrm{Tr}_{\mathcal{H}_1}\rho\big) - E_f(\rho) : \rho \text{ state on } \mathcal{K}\}. \quad (3)$$

Theorem 1 *If for any two channels T and T', with fixed Stinespring dilation as above, $C(T\otimes T') = C(T)+C(T')$, then*

$$E_f(\rho_T \otimes \rho_{T'}) = E_f(\rho_T) + E_f(\rho_{T'}),$$

where ρ_T is a state which maximise eq. (3). □

Most interesting is the case when we know $C(T^{\otimes n}) = nC(T)$, such as (I)-(III), for which we can conclude

$$E_f(\rho_T^{\otimes n}) = nE_f(\rho_T) = nE_c(\rho_T). \quad (4)$$

4 Group symmetry

Making use of group symmetries, like [14], we can prove the additivity of E_f for more states. The examples (I) - (II) in previous section satisfy the following. A compact group G acts irreducibly both on $\mathcal{K}$ and $\mathcal{H}_2$ by a unitary representation (which we denote by V_g and U_g), which commutes with the map T (partial trace):

$$\mathrm{Tr}_{\mathcal{H}_1}\big(V_g\sigma V_g^\dagger\big) = U_g\big(\mathrm{Tr}_{\mathcal{H}_1}\sigma\big)U_g^\dagger. \quad (5)$$

By use of this symmetry and eq. (4), for *all* states ρ spanned by $\{V_g|\psi_0\rangle\langle\psi_0|V_g^* : g \in G\}$, where $|\psi_0\rangle$ is a pure state with $E_f(|\psi_0\rangle) = \min\{E_f(|\psi\rangle) : |\psi\rangle \in \mathcal{K}\}$, we can conclude,

$$E_c(\rho) = E_f(\rho) = \min\{E(\psi) : |\psi\rangle \in \mathcal{K}\}. \quad (6)$$

If in addition the action of G in $\mathcal{K}$ is *transitive* like in the example (II), eq. (6) holds for all the state supported on $\mathcal{K}$, because $E_f(|\psi\rangle)$ takes the same value for any pure state $|\psi\rangle$ in $\mathcal{K}$.

5 Gap between E_c and E_D

The distillable entanglement, $E_D(\rho)$ measures the number of bell pairs which is distillable from infinitely many copies of ρ, which, in case that ρ is a pure state, equals $E_c(\rho) = S(\mathrm{Tr}_{\mathcal{H}_\infty}\rho)$. In case that ρ is a mixed state, in general, $E_D(\rho) \leq E_c(\rho)$, and, in some cases, strict inequality holds [15]. In our example (I), we can supply some more examples of such states by use of the inequality [15],

$$\log \|\rho^\Gamma\|_1 > E_D(\rho).$$

By use of the discussion in previous section, we have,

$$E_c(\rho_{T,s}) = H(p_0 + p_z, p_x + p_y),$$

where

$$\begin{aligned} \rho_{T,s} &= s|\psi_T\rangle\langle\psi_T| + (1-s)|\psi_T^\perp\rangle\langle\psi_T^\perp|, \\ |\psi_T\rangle &= \sqrt{p_0}|0\rangle \otimes |0\rangle + \sqrt{p_x}|1\rangle \otimes |x\rangle \\ &\quad + i\sqrt{p_y}|1\rangle \otimes |y\rangle + \sqrt{p_z}|0\rangle \otimes |z\rangle, \\ |\psi_T^\perp\rangle &= \sqrt{p_0}|1\rangle \otimes |0\rangle + \sqrt{p_x}|0\rangle \otimes |x\rangle \\ &\quad - i\sqrt{p_y}|0\rangle \otimes |y\rangle - \sqrt{p_z}|1\rangle \otimes |z\rangle, \end{aligned}$$

with

$$p_0 + p_z - p_x - p_y \geq |p_0 + p_y - p_x - p_z|, |p_0 + p_x - p_y - p_z|. \quad (7)$$

By some elementary considerations, $\log \|\rho^\Gamma_{T,\frac{1}{2}}\|_1 < E_c(\rho_T)$ is equivalent to,

$$\begin{aligned} &z^4 - z^3 + 4(p_0p_xp_y + p_0p_xp_z + p_0p_yp_z + p_xp_yp_z)z \\ &\quad - 16p_0p_xp_yp_z > 0, \end{aligned} \quad (8)$$

with $z = -\frac{2^{E_c(\rho_{T,\frac{1}{2}})} - 1}{2}$ (figure 1). Therefore, in this region,

$$E_D(\rho_{T,\frac{1}{2}}) < E_c(\rho_{T,\frac{1}{2}}).$$

If $p_0 + p_z = p_x + p_y = \frac{1}{2}$ and $p_0 \neq p_z$, $p_x \neq p_y$, we can prove the gap even for *all* $0 < s < 1$,

$$E_D(\rho_{T,s}) < E_c(\rho_{T,s}).$$

6 Superadditivity of E_f?

Conjecture 2 *(Superadditivity) Let ρ be a state on $\mathcal{H} \otimes \mathcal{H}'$, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}' = \mathcal{H}'_1 \otimes \mathcal{H}'_2$. Then,*

$$E_f(\rho) \geq E_f(\mathrm{Tr}_{\mathcal{H}'}\rho) + E_f(\mathrm{Tr}_{\mathcal{H}}\rho), \quad (9)$$

where all entanglements of formation are understood with respect to the 1–2–partition of the respective system.

If this superadditivity conjecture turns out to be true, additivity both of E_f and of Holevo capacity will be obtain as its corollaries.

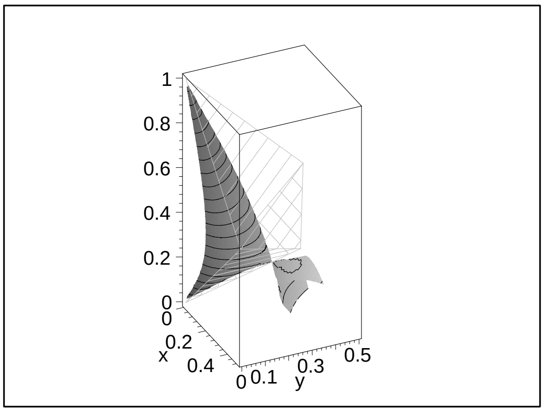

Figure 1: Plots in a (p_x, p_y, p_z)–frame of the admissible parameters according to eq. (7) and of the region for which eq. (8)holds (between the two surfaces).

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