Size of the separable neighborhood of the maximally mixed bipartite quantum state

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For finite-dimensional bipartite quantum systems, we find the size of the largest balls, in spectral l_p norms for $1 \leq p \leq \infty$, of separable (unentangled) matrices around the identity matrix.

Determining whether a given state, even of two quantum systems, is entangled or separable (not entangled) is in general a difficult question, and considerable effort has been expended on finding necesssary and/or sufficient conditions. It is known that there is a ball of separable states centered at the maximally mixed or "uniform" density matrix (the normalized identity matrix), and the size of this ball is of theoretical and practical interest. For example, the "pseudopure" states which describe the state of each molecule in NMR quantum information processing consist of mixtures of the uniform density matrix with a large coefficient, and a small admixture of a pure state which gives the signal that the desired quantum dynamics is occuring. A corollary of the main result of this paper gives bounds of $1/N^2$ and $2/(2+N^2)$ on the fraction of a mixture which can be nonuniform and still be guaranteed to give a separable state (for a system composed of two N-dimensional subsystems).

Our main result is that the matrix $I + \Delta$ is separable for all hermitian Δ with $||\Delta||_2 \leq 1$.

We represent unnormalized states of a bipartite quantum system composed of two subsystems of dimensions M and N, as positive semidefinite $M \times M$ block matrices, with $N \times N$ blocks, giving the matrix elements of the density operator in some fixed tensor product basis.

Positive linear maps $\phi : M_m \rightarrow M_n$ are those linear maps from $m \times m$ matrices to $n \times n$ matrices, which preserve positive semidefiniteness. They are Hermitian-preserving. Such a map ϕ is *stochastic* if $\phi(I) = I$. Applying it to the N-dimensional subsystem while doing nothing to the other system is just applying ϕ to each block of X; we call the resulting map on the matrices of the bipartite system $\tilde{\phi}$. A basic necessary and sufficient condition for separability of A is that for any positive linear map ϕ, $\tilde{\phi}(A)$ be positive semidefinite [4], [3]. In fact, stochastic positive maps suffice. For qubits, the partial transposition map ("Woronowicz-Peres (WP) criterion") suffices.

The proof of the main result of this paper uses two propositions.

Proposition 1: If $||X|| \leq 1$, the block matrix

$$\begin{pmatrix} I & X \\ X^\dagger & I \end{pmatrix} \tag{1}$$

is separable. This is a special case (M=2) of a recent theorem [5] that all positive semidefinite $M \times M$ block Toeplitz or block Hankel matrices whose blocks are $N \times N$ matrices are separable. [5] also contains two alternative proofs for the special case $M = 2$.

We can use Proposition 1 to show a contraction inequality (Proposition 2 below).

Proposition 2: Let $\phi : M_n \rightarrow M_n$ be a stochastic positive linear map. Then for any $X \in M_n$, $||\phi(X)|| \leq ||X||$.

Proof 1: We show that $\phi(X) \leq 1$ if $||X|| \leq 1$; the proposition follows by ϕ's linearity. Apply $\tilde{\phi}$ to (1) obtaining:

$$\begin{pmatrix} I & \phi(X) \\ \phi(X^\dagger) & I \end{pmatrix} = \begin{pmatrix} I & \phi(X) \\ \phi(X)^\dagger & I \end{pmatrix} . \tag{2}$$

Since this resulted from applying $\tilde{\phi}$ to a separable state, it is positive semidefinite. Hence, $\phi(X)^\dagger \phi(X) \leq I$, so $||X|| \leq 1$.

This proof was independently discovered in [6].

We proceed to the main theorems.

Theorem 1: The matrix $I + \Delta$ is separable for all hermitian Δ with $||\Delta||_2 \leq 1$.

Proof:

$$||\tilde{\phi}(\Delta)||^2 \leq ||A||^2 \leq ||A||_2^2, \tag{3}$$

where $A := [a_{ij}]$, $a_{ij} := ||\phi(\Delta_{ij})||$.

$$||A||_2^2 = \sum_{ij} a_{ij}^2 = \sum_{ij} ||\phi(\Delta_{ij})||^2. \tag{4}$$

(The first inequality is because the operator norm of a block matrix is bounded above by that of the matrix whose elements are the norms of the blocks, and the second is because the Frobenius norm is an upper bound to the operator norm.) But $||\phi(\Delta_{ij})||^2 \leq ||\Delta_{ij}||$ by Prop. 2, and this in turn is less than $||\Delta_{ij}||_2$. So

$$||\tilde{\phi}(\Delta)||^2 \leq \sum_{ij} ||\phi(\Delta_{ij})||^2 \leq \sum_{ij} ||\Delta_{ij}||_2^2 \equiv ||\Delta||_2^2 \leq 1, \tag{5}$$

the last inequality being the premise of the theorem. Having shown that $||\tilde{\phi}(\Delta)|| \leq 1$, separability of $I + \Delta$ follows by the WP criterion.

Let us now present some corollaries of Theorem 1.

Define $||\Delta||_p := (\sum_i |\lambda_i|^p)^{\frac{1}{p}}$ (λ_i being the eigenvalues of the square hermitian matrix Δ .)

Corollary 1 (l_p balls): Consider $N \times N$ system.Then the matrix $I+\Delta$ is separable for all hermitian Δ with $||\Delta||_p \leq 1 (1 \leq p \leq 2)$ and $||\Delta||_p \leq B(N,p) =: N^{\frac{2}{p}-1} (2 \leq p \leq \infty)$. The proof uses elementary inequalities between l_r and l_s

norms ($1 \leq r, s \leq \infty$).
The l_∞ result was also obtained by different methods in [6]. The p-balls in Cor. 1 are clearly the largest possible for $1 \leq p \leq 2$. What about $2 < p \leq \infty$?
Theorem 2: 1. Consider a pure ρ corresponding to a state $|\psi\rangle = \sum_{ij} \psi_{ij} e_i \otimes e_j$, $1 \leq i, j \leq N$. The spectrum of ρ^T is $(d_1^2, ..., d_N^2; d_i d_j, -d_i d_j (1 \leq i \neq j \leq N))$, where $d_1, ..., d_N$ are the singular values of the $N \times N$ matrix $\psi := [\psi_{ij} : 1 \leq i, j \leq N]$.
2. Define

$$W(N) = \max_{\rho \in Den(N,N)} -\lambda_{min}(\rho^T),$$

where $Den(N, N)$ stands for the compact convex set of all density matrices of $N \times N$ bipartite quantum systems. Then $W(N) = \frac{1}{2}$.
3. If $I + a\rho$ is separable for all ρ and $a > 0$ then $a \leq W(N)^{-1} = 2$.
Proving 1. involves transforming ρ to the Schmidt basis and calculating the spectrum of the transpose, which is the same as ρ^T's. Part 2's bound uses $2ab \leq a^2 + b^2$ and is achieved by $(1/\sqrt{2}, 1/\sqrt{2}, 0, ..., 0)$. Part 3 is immediate from the WP criterion.

Contrary to the "folklore," in the result above the fully entangled state is not the worst one; instead the worst one is a maximally entangled state of two local two-dimensional subspaces.

Before formulating the next result let us introduce some notation. We write $WP(N, N)$ for the closed convex cone of (unnormalized) $N^2 \times N^2$ density matrices with positive partial transpose. $Sep(N, N)$ is the closed convex cone of (unnormalized) separable Hermitian matrices. $WP(N, N) \subset Sep(N, N)$.
Theorem 3: Suppose that $p > 1$. If the p-ball $Ball(N, p, a) = \{A \in H(N^2) : A = I + \Delta, \ ||\Delta||_p \leq a\}$ belongs to $WP(N, N)$ then $a \leq B(N, p) =: N^{-1+\frac{2}{p}} (2 \leq p \leq \infty)$.
As $WP(N, N) \subset Sep(N, N)$ this Theorem proves that the l_p-balls $Ball(N, p, B(N, p))$ in Corollary 1 are largest possible.
Corollary 2 (scaling) : Let A be an (unnormalized) density matrix of a bipartite system with total dimension $d = NM$ and $\lambda = (\lambda_1, ..., \lambda_d)$ be the vector of eigenvalues of A. If

$$S(\lambda) =: d - \frac{||\lambda||_1^2}{||\lambda||_2^2} \leq 1 \tag{6}$$

then A is separable.
Proof: It is easy to see that $S(\lambda) = \min_{a>0} ||a\lambda - e||_2^2$, where e is a vector of all ones. Therefore if $S(\lambda) \leq 1$ then $A = b(I + \Delta)$, where $b > 0$ and $||\Delta||_2^2 \leq 1$. It follows from Theorem 1 that $A = b(I + \Delta)$ is separable.
Corollary 3 (largest Frobenius ball for density matrices): Suppose that A is a normalized density matrix of a bipartite system with total dimension $d = NM$, i.e. $\sum_{1 \leq i \leq d} \lambda_i = 1$ and $\lambda_i \geq 0, 1 \leq i \leq d$. If $||A - \frac{1}{d} I||_2^2 = ||\lambda - \frac{1}{d} e||_2^2 \leq \frac{1}{d(d-1)} =: r^2$ then A is separable. r is the largest such constant.
Remark: In terms of the "purity" tr ρ^2 of the density matrix (which takes the value 1 for pure states and $1/d$ for the maximally mixed state), Corollary 3 says that ρ is separable if its purity is less than or equal to $1/(d-1)$.

Rungta *et. al.* [2], extending the methods of [1], considered a tensor product of R systems, each N-dimensional, in mixtures $\rho = (1 - \epsilon) I / N^R + \epsilon \rho'$ with ρ' a normalized density matrix. They found lower and upper bounds on the size ϵ_{max} of the largest ϵ-ball for which these states can be guaranteed to be separable: $1/(1 + N^{2R-1}) \leq \epsilon_{max} < 1/(1 + N^{R-1})$. (The $N = 2$ case is in [1].)

For $R = 2$ (bipartite states), the bounds of [2] are $1/(1 + N^3)$ and $1/(1 + N)$. The lower bound is close to what one can get from the l_∞ (operator norm) result, while the upper bound comes from mixing in ρ_N. Our results give $1/N^2 \leq \epsilon_{max} \leq 2/(2+N^2)$. The lower bound is via Theorem 1, with the added tightness due to the use of Frobenius rather than operator norm, while the upper bound comes from Theorem 2 and the discussion preceding it (and is tighter because ρ_N is not the optimal pure state to mix in).

For multipartite states ($R > 2$) and the obvious generalization of the notion of separability, we get a slightly better upper bound than [2]. For example, for even R, $\epsilon_{max} \leq 2/(2+N^R)$, by viewing the state as bipartite with two equal-size sets of qubits as subsystems. It is only a bound because a state could be separable with respect to every bipartition of the subsystems yet not be separable (such states are known [8]).

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