A LIMIT THEOREM FOR THE HADAMARD WALK IN ONE DIMENSION

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We present a new type of limit theorems for the Hadamard walk. In contrast with the de Moivre-Laplace limit theorem, our symmetric case implies that X_n^{φ}/n converges in distribution to a limit Z^{φ} as $n \to \infty$ where Z^{φ} has a density $1/\pi(1-x^2)\sqrt{1-2x^2}$ for $x \in (-\sqrt{2}/2, \sqrt{2}/2)$.

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The classical symmetric random walk on the line is the motion of a particle which inhabits the set of integers. The particle moves at each step either one step to the right with probability 1/2 or one step to the left with probability 1/2. The directions of different steps are independent of each other. Here we consider quantum variations of the above classical random walk. Very recently quantum random walks have been widely investigated by a number of groups in connection with the quantum computing, for examples, [1-13]. For more general setting including quantum cellular automata, see [14]. In [2], they gave two general ideas for analyzing quantum random walks. One is the path integral approach, the other is the Schrödinger approach. Here we take the path integral approach, that is, the probability amplitude of a state for the quantum random walk is given as a combinatorial sum over all possible paths leading to that state. In this paper we focus on the Hadamard walk which has been extensively investigated in the study of quantum random walks. The time evolution of the one-dimensional Hadamard walk studied here is given by the following unitary matrix (see [15]):

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard walk is a quantum generalization of the classical symmetric random walk in one dimension with an additional degree of freedom called the chirality. However the symmetry of the Hadamard walk depends heavily on initial qubit state, see [9]. The chirality takes values left and right, and means the direction of the motion of the particle. The evolution of the quantum random walk is given by the following way. At each time step, if the particle has the left chirality, it moves one step to the left, and if it has the right chirality, it moves one step to the right.

More precisely, the Hadamard matrix H acts on two chirality states $|L\rangle$ and $|R\rangle$: $|L\rangle \to (|L\rangle + |R\rangle)/\sqrt{2}$, $|R\rangle \to (|L\rangle - |R\rangle)/\sqrt{2}$ where L and R refer to the right and left chirality state respectively. In fact, define $|L\rangle = {}^t[1,0]$, $|R\rangle = {}^t[0,1]$, so we have $H|L\rangle \to (|L\rangle + |R\rangle)/\sqrt{2}$, $H|R\rangle \to (|L\rangle - |R\rangle)/\sqrt{2}$ where t means the trasposed operator. We introduce P and Q matrices as follows:

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

with $H = P + Q$. Here P (resp. Q) represents that the particle moves to the left (resp. right) with equal probability. We should remark that P and Q are useful tools in the study of the iterates of H. However, they cannot be interpreted as dynamical evolution operators since they are not unitary. In the present paper, the study on the dependence of a limit distribution on initial qubit state is one of the essential parts, so we define the set of initial qubit states as follows:

$$\Phi = \left\{ \varphi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbf{C}^2 : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Let X_n^{φ} be the Hadamard walk at time n starting from initial qubit state $\varphi \in \Phi$. It is noted that $X_0^{\varphi} = 0$. For fixed l and m with $l + m = n$ and $m - l = k$,

$$\Xi(l,m) = \sum_{l_j, m_j} P^{l_1} Q^{m_1} P^{l_2} Q^{m_2} \cdots P^{l_n} Q^{m_n}$$

summed over all $l_j, m_j \ge 0$ satisfying $m_1 + \cdots + m_n = m$ and $l_1 + \cdots + l_n = l$. Moreover, to define $P(X_n^{\varphi} = k)$, it is convenient to introduce

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

We should remark that the set of $P, Q, R,$ and S forms an orthonormal basis of the vector space of complex 2×2 matrices. An expression of $\Xi(l,m)$ can be given by using P, Q, R and S (see Theorem 3 in [9]). The definition of $\Xi(l,m)$ gives

$$P(X_n^{\varphi} = k) = (\Xi(l,m)\varphi)^* (\Xi(l,m)\varphi)$$

where $n = l + m$ and $k = -l + m$. From this expression, we obtain the characteristic function of X_n^{φ} and the mth moment of it. One of the interesting results is that when

m is even, the mth moment of X_n^φ is independent of the initial qubit state $\varphi \in \Phi$. On the other hand, when m is odd, the mth moment depends on the initial qubit state. So the standard deviation of X_n^φ in not independent of the initial qubit state $\varphi \in \Phi$. The above mentioned results for general 2×2 unitary matrices appeared in [8].

For the Hadamard walk, we have the following new type of limit theorems: if $-\sqrt{2}/2 < a < b < \sqrt{2}/2$, then as $n \to \infty$,

$$P(a \le X_n^\varphi/n \le b) \to \int_a^b \frac{1 - (|\alpha|^2 - |\beta|^2 + \alpha\overline{\beta} + \overline{\alpha}\beta)x}{\pi(1-x^2)\sqrt{1-2x^2}} dx$$

for any initial qubit state $\varphi = {}^t[\alpha, \beta]$. The above density function is denoted by $f(x;\varphi)$. For the classical symmetric random walk Y_n^o starting from the origin, the well-known central limit theorem implies that if $-\infty < a < b < \infty$, then as $n \to \infty$,

$$P(a \le Y_n^o/\sqrt{n} \le b) \to \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

This result is often called the de Moivre-Laplace limit theorem. When we take $\varphi = {}^t[1/\sqrt{2}, i/\sqrt{2}]$ (symmetric case), then we have the following quantum version of the de Moivre-Laplace limit theorem: if $-\sqrt{2}/2 < a < b < \sqrt{2}/2$, then as $n \to \infty$,

$$P(a \le X_n^\varphi/n \le b) \to \int_a^b \frac{1}{\pi(1-x^2)\sqrt{1-2x^2}} dx$$

So there is a remarkable difference between the quantum random walk X_n^φ and the classical one Y_n^o even in a symmetric case for $\varphi = {}^t[1/\sqrt{2}, i/\sqrt{2}]$. The above limit theorem can be also extended to $\varphi \in \Phi_\perp$, since Konno, Namiki and Soshi [9] gave $\Phi_\perp = \Phi_s = \Phi_0$ where

$$\begin{aligned} \Phi_\perp &= \left\{ \varphi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \Phi : |\alpha| = |\beta|,\ \alpha\overline{\beta} + \overline{\alpha}\beta = 0 \right\} \\ \Phi_s &= \{ \varphi \in \Phi :\ P(X_n^\varphi = k) = P(X_n^\varphi = -k) \\ &\qquad\qquad \text{for any } n \in \mathbf{Z}_+ \text{ and } k \in \mathbf{Z} \} \\ \Phi_0 &= \{ \varphi \in \Phi :\ E(X_n^\varphi) = 0 \ \text{for any } n \in \mathbf{Z}_+ \} \end{aligned}$$

and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$) is the set of (resp. non-negative) integers. Noting that $E(X_n^\varphi) = 0$ $(n \ge 0)$ for any $\varphi \in \Phi_\perp$, we have

$$V(X_n^\varphi)/n^2 \to (2 - \sqrt{2})/2 = 0.29289\ldots$$

where $V(X)$ is the variance of X. So the standard deviation of the limit distribution is given by $\sqrt{(2-\sqrt{2})/2} = 0.54119\ldots$. This rigorous result reveals that numerical simulation result $3/5 = 0.6$ given by [12] is not so accurate. As in a similar way, when we take $\varphi = {}^t[0, e^{i\theta}]$ where $\theta \in [0, 2\pi)$ (asymmetric case), we see that if $-\sqrt{2}/2 < a < b < \sqrt{2}/2$, then as $n \to \infty$,

$$P(a \le X_n^\varphi/n \le b) \to \int_a^b \frac{1}{\pi(1-x)\sqrt{1-2x^2}} dx$$

So we have

$$\begin{aligned} E(X_n^\varphi)/n &\to (2-\sqrt{2})/2 = 0.29289\ldots \\ V(X_n^\varphi)/n^2 &\to (\sqrt{2}-1)/2 = 0.20710\ldots \end{aligned}$$

When $\varphi = {}^t[0,1]$ $(\theta = 0)$, Ambainis *et al.* [2] gave the same result. In their paper two approaches are taken, that is, the Schrödinger approach and the path integral approach. However their result comes mainly from the Schrödinger approach by using a Fourier analysis. The details on the derivation based on the path integral approach is not so clear compared with [8]. In another asymmetric case $\varphi = {}^t[e^{i\theta}, 0]$ where $\theta \in [0, 2\pi)$, a similar argument implies that if $-\sqrt{2}/2 < a < b < \sqrt{2}/2$, then as $n \to \infty$,

$$P(a \le X_n^\varphi/n \le b) \to \int_a^b \frac{1}{\pi(1+x)\sqrt{1-2x^2}} dx$$

Noting that $f(-x; {}^t[e^{i\theta}, 0]) = f(x; {}^t[0, e^{i\theta}])$ for any $x \in (-\sqrt{2}/2, \sqrt{2}/2)$, we have the following same results as in the previous case $\varphi = {}^t[0, e^{i\theta}]$. So the standard deviation of the limit distribution is given by $\sqrt{(\sqrt{2}-1)/2} = 0.45508\ldots$. Simulation result 0.4544 ± 0.0012 in [10] (their case is $\theta = 0$) is consistent with our rigorous result.

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