One-dimensional quantum walks with absorbing boundaries

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Several recent papers have studied the properties of *quantum walks*, which are quantum computational variants of discrete-time random walks . The behavior of quantum walks differs from that of ordinary random walks in several striking ways, due to the fact that quantum walks exhibit interference patterns whereas ordinary random walks do not. For instance, mixing times, hitting times, and exit probabilities of quantum walks can differ significantly from analogously defined random walks.

Ordinary random walks have had many applications in computer science, particularly as algorithmic tools. Examples include randomized algorithms for graph connectivity, 2SAT, and approximating the permanent. Quantum walks have the potential to offer new tools for the design of quantum algorithms, which is one of the primary motivations for studying their behavior.

In this paper we analyze the behavior of quantum random walks. In particular we present several new results for the absorption probabilities in systems with both one and two absorbing walls for the one-dimensional case. We compute these probabilities both by employing generating functions and by use of an eigenfunction approach. The generating function method is used to determine some simple properties of the walks we consider, but appears to have limitations. The eigenfunction approach works by relating the problem of absorption to a unitary problem that has identical dynamics inside a certain domain, and can be used to compute several additional interesting properties, such as the time dependence of absorption. The eigenfunction method has the distinct advantage that it can be extended to arbitrary dimensionality. We outline the solution of the absorption probability problem of a (d-1)-dimensional wall in a d-dimensional space.

More precisely, our results may be summarized as follows. First, for one-barrier systems, we obtain exact expressions for the probability of absorption by the barrier, as a function of the initial distance to the barrier. (Complementing these formulas gives the probability of escape to infinity.) These expressions involve integrals of different forms coming from our two methods of analysis. Both forms allow asymptotic analysis of the absorption probabilities; in particular, we compute the limiting probabilities when the initial distance to the barrier is large. We do this both for Hadamard walks and for walks based on more general unitary transformations. Next, for the two-barrier Hadamard system, we compute the long-time limit of the probability of absorption by each barrier when the walker starts off very far from one barrier but an arbitrary distance from the other barrier. Again, the expressions involve integrals whose asymptotic limits are easily analyzed, so that we can compute the limiting probabilities when the initial distance to both barriers is large. We then outline how the eigenfunction method can be used to analyze the behavior in small systems. Then we use the eigenfunction method to analyze the time dependence of the absorption in the limit of long times for walks with both one and two walls. We find that the approach to the asymptotic limit is much slower when there are two walls, where the probability remaining to be absorbed at time t decays as $1/\sqrt{t}$, than when the system has one wall, where the probability remaining to be absorbed at time t is proportional to $1/t^2$. Finally, we study d-dimensional walks, and show that the region over which most of the probability is distributed by time t has volume proportional to t^d. We indicate how to solve the problem of the absorption of a $(d-1)$-dimensional wall without giving explicit results.

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