Error exponents for entanglement concentration

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Asymptotic entanglement concentration is discussed. We give the distillable entanglement as a function of an error exponent. The formula fills the gap between the least upper bound of distillable entanglement in probabilistic concentration, which is the well-known entropy of entanglement, and the maximum attained in deterministic concentration. A strong converse of entanglement concentration is also presented.

Quantum entanglement, an indispensable resource for quantum information processing is expected to have a rich mathematical structure behind its weirdness. As in the case of other physical resources, quantification of entanglement is the key to understanding its full potential. The fundamental results are the intimate connection between the mathematical theory of majorization and entanglement manipulation [1–4], and the existence of a unique measure of entanglement in the asymptotic limit [5, 6]. Entanglement concentration has been discussed extensively as one way of quantifying pure-state entanglement [2–5, 7–10], but here we deal with it from the viewpoint of error exponents.

Suppose we share n identical copies of a partially entangled state $|\phi\rangle = \sum_{i=1}^{d} \sqrt{p_i}|i\rangle|i\rangle$, where the Schmidt coefficients squared are arranged in decreasing order, i.e., $p_1 \geq p_2 \geq \cdots \geq p_d \geq 0$, and sum to one. Consider entanglement concentration that converts $|\phi\rangle^{\otimes n}$ into a maximally entangled state of size L_n with the optimal success probability P_{L_n}. Bennett *et al.* [5] proved that the maximum number of Bell pairs distilled per copy from $|\phi\rangle^{\otimes n}$ is given by

$$E_{\mathrm{entropy}}(\phi) = -\sum_{i=1}^{d} p_i \log_2 p_i, \qquad (1)$$

in the asymptotic limit, $n \to \infty$. (Logarithms are taken to base two throughout this paper.) They imposed the condition that the success probability of entanglement concentration tends to one in the asymptotic limit, i.e., $P_{L_n} = 1 - \epsilon$, where $\epsilon \to 0$ as $n \to \infty$.

On the other hand, the maximum entanglement yield in deterministic concentration [9] becomes

$$E_{\mathrm{det}}(\phi) = -\log p_1. \qquad (2)$$

The restriction *deterministic* means that the process succeeds with probability one both in finite regimes and in the asymptotic limit.

Though the quantities E_{entropy} and E_{det} give entanglement yield in the asymptotic limit, where both processes succeed with probability one, the two quantities do not coincide. We will see that the discrepancy is caused by the difference of the rate at which failure probabilities decrease when n tends to infinity in both concentration processes.

In the following, we discuss the case where the optimal success probability P_{L_n} converges to one as the number of entangled pairs n increases. The rate of the convergence is represented by an error exponent r, the first order coefficient in the exponent of the failure probability in the asymptotic limit, which is defined as

$$r = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log(1 - P_{L_n}) \right\}. \qquad (3)$$

We will present the maximum number of Bell pairs distilled per copy in the asymptotic limit, E, as a function of the error exponent r by using the Shannon entropy $H(p) = -\sum_{i=1}^{d} p_i \log p_i$ and the relative entropy $D(p \parallel q) = \sum_{i=1}^{d} p_i \log \frac{p_i}{q_i}$, where p and q are probability distributions. First, we present a theorem that relates entanglement yield and an error exponent via a monotone function, from which we will derive a formula for entanglement yield $E(r)$.

Theorem 1 *Consider a sequence of entanglement concentration schemes converting n identical copies of* $|\phi\rangle = \sum_{i=1}^{d} \sqrt{p_i}|i\rangle|i\rangle$, *i.e.,* $|\phi\rangle^{\otimes n}$, *into a maximally entangled state of size* L_n, *which attain the optimal success probability* P_{L_n}. *Suppose*

$$\limsup_{n\to\infty} \left(\frac{1}{n}\log L_n\right) < H(p), \tag{4}$$

and

$$\frac{1}{n}\log L_n > -\log p_1, \tag{5}$$

where $p = (p_1, \cdots, p_d)$. *Then,*

$$\limsup_{n\to\infty} \left(\frac{1}{n}\log L_n\right) = f\left(\liminf_{n\to\infty} \left\{-\frac{1}{n}\log(1-P_{L_n})\right\}\right), \tag{6}$$

and

$$\liminf_{n\to\infty} \left(\frac{1}{n}\log L_n\right) = f\left(\limsup_{n\to\infty} \left\{-\frac{1}{n}\log(1-P_{L_n})\right\}\right), \tag{7}$$

where $f(r) \equiv \min_{q:D(q\|p)\le r} \{D(q \parallel p) + H(q)\}$.

Theorem 1 leads to the following corollary, which gives the maximum asymptotic entanglement yield $E(r)$ under the requirement that the failure probability decreases as rapidly as 2^{-nr}:

Corollary 2 *Consider a sequence of entanglement concentration schemes converting* $|\phi\rangle^{\otimes n}$ *into a maximally entangled state of size* L_n *with success probability* $P^{(n)}_{success}$, *such that*

$$r \le \liminf_{n\to\infty} \left\{-\frac{1}{n}\log(1-P^{(n)}_{success})\right\}. \tag{8}$$

Let us denote the class of all such sequences by $\mathcal{C}(r)$. *Then, for* $r > 0$,

$$\begin{aligned} E(r) &\equiv \max_{\mathcal{C}(r)} \limsup_{n\to\infty} \left(\frac{1}{n}\log L_n\right) = \max_{\mathcal{C}(r)} \liminf_{n\to\infty} \left(\frac{1}{n}\log L_n\right) \\ &= \min_{q:D(q\|p)\le r} \{D(q \parallel p) + H(q)\}. \end{aligned} \tag{9}$$

This corollary provides the missing link between the least upper bound of distillable entanglement in probabilistic concentration and the maximum attained in deterministic one: $E_{\text{entropy}} = H(p) = \lim_{r\to 0} E(r)$ and $E_{\text{det}} = -\log p_1 = \lim_{r\to\infty} E(r)$.

We have discussed how entanglement yield behaves when the failure probability exponentially decreases. The above results are obtained by using the method of types [11, 12]. In addition, we consider the case where the *success* probability exponentially decreases. Assuming that the optimal success probability converges to zero as the number of entangled pairs n increases, we can also derive the distillable entanglement as a function of the exponent of the success probability in a similar way to the asymptotically successful case mentioned above. The analysis shows that the success probability exponentially decreases when we try to distill more entanglement than $H(p)$ (strong converse). This was observed in Ref. [7], but here we are able to drive the *exact* error rate.

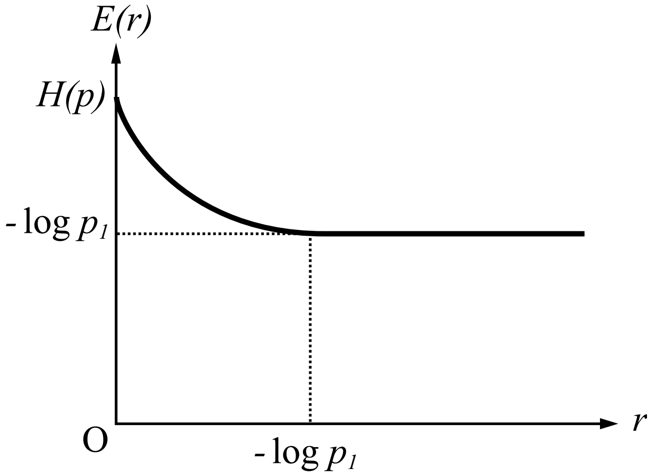

FIG. 1: Entanglement yield in asymptotic entanglement concentration with an error exponent r. The horizontal axis represents the error exponent. The vertical axis represents the number of Bell pairs distilled per copy in the asymptotic limit: $E(r) = \min_{q:D(q\|p)\le r} \{D(q \parallel p) + H(q)\}$.

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