Local properties of the closest disentangled and PPT states

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We consider the local filtering in order to investigate the properties of the closest disentangled and PPT states in any system (multi-party with any dimension). This physical operation ensures that the state after the operation is disentangled (or PPT) if the state before operation is disentangled (or PPT). As a result, we can obtain some equations the closest disentangled and PPT state must satisfy. Further, we find some sufficient conditions for which the closest disentangled (or PPT) state has the same reduction as the given entangled state.

Quantum entanglement is the most striking feature of quantum mechanics. In order to quantify the resource of the entanglement, several measures have been proposed. The relative entropy of entanglement [1,2] is defined as the distance to the disentangled state closest to the given entangled state under the measure of the relative entropy. This implies that the closest disentangled state plays an important role to quantify the quantum entanglement. In addition, the closest disentangled state itself answers the following question: What is the state when the quantum correlation is completely but minimally (maintaining the classical correlation as long as possible [2]) washed out? Therefore, it will be important to clarify the properties of the closest disentangled state itself to understand the characteristics of the quantum entanglement.

Further, the analytical formula of the relative entropy of entanglement have been strongly desired to clarify the relations between the entanglement and the performance of many applications of quantum information. However, deriving the analytical formula has been known to be a hard problem even in the simplest two-qubit system. Mathematically, the difficulty lies in searching for the closest disentangled state on the complicated boundary surface of the set of disentangled states in the Hilbert space. Therefore, to investigate the closest disentangled state might be also important in a sense that it might give some hints for solving the hard problem.

In this paper, we consider the physical operation of the local filtering in order to investigate the properties of the closest disentangled states. This physical operation ensures that the state after the operation is disentangled if the state before operation is disentangled. As a result, we can obtain some equations the closest disentangled state must satisfy, in spite that the geometry of the entangled-disentangled boundary is quite complicated. In particular, we show that the reduction of the closest disentangled state is strongly related to the extremal condition of the local filtering on each party. Although the equations we obtain are not still tractable, we find some sufficient conditions for which the closest disentangled state has the same reduction as the given entangled state.

For a given entangled state ϱ, its relative entropy of entanglement is defined as [1,2]

$$E_R(\varrho) = \min_{\sigma\in\mathcal{D}} S(\varrho||\sigma) = \min_{\sigma\in\mathcal{D}} \Big[\mathrm{Tr}\varrho\log\varrho - \mathrm{Tr}\varrho\log\sigma\Big], \quad (1)$$

where the minimization is performed over the set of disentangled states $\mathcal{D}$. Let us assume that σ^* is the closest disentangled state which minimizes $S(\varrho||\sigma)$, and hence

$$S(\varrho||\sigma) \geq S(\varrho||\sigma^*) \quad (2)$$

for any $\sigma \in \mathcal{D}$. Among those disentangled states, we consider the state σ' which is obtained from σ^* by local filtering operations. It should be noted that, in the definition of the relative entropy of entanglement, the set of $\mathcal{D}$ is sometimes taken for the positive partial transposed (PPT) states [3], and the state σ^* achieving the minimum is the closest PPT state. Even in this case, σ' obtained from σ^* by local filtering is also PPT. Therefore, all the results for the closest disentangled states shown below also hold for the closest PPT states.

Let us consider Bob's local filtering in two qubits:

$$\sigma' = \frac{(I\otimes e^{t\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I\otimes e^{t\vec{n}\cdot\vec{\sigma}/2})}{\mathrm{Tr}[(I\otimes e^{t\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I\otimes e^{t\vec{n}\cdot\vec{\sigma}/2})]}, \quad (3)$$

where $\vec{\sigma}$ is the vector of Pauli matrices, and t is a real parameter. Using the polynomial expansion with respect to t, we obtain

$$\begin{aligned}\mathrm{Tr}\varrho\log\sigma' = {} & \mathrm{Tr}\varrho\log\sigma^* \\ & + t\Big[\mathrm{Tr}\varrho\int_0^\infty \frac{1}{\sigma^*+x}\frac{\{\sigma^*,(I\otimes\vec{n}\cdot\vec{\sigma})\}}{2}\frac{1}{\sigma^*+x}dx \\ & -\mathrm{Tr}[(I\otimes\vec{n}\cdot\vec{\sigma})\sigma^*]\Big] + \mathcal{O}(t^2), \quad (4)\end{aligned}$$

where $\{A,B\}\equiv AB+BA$. If the linear coefficient of t is not zero, there always exists σ' satisfying $S(\varrho||\sigma') < S(\varrho||\sigma^*)$ for a small enough $|t|$, but this contradicts Eq. (2). Therefore the linear coefficient must be zero for any direction of $\vec{n}$. Then σ^* must satisfy

$$\vec{s}_B = \vec{r}_B + \vec{g}_B, \quad (5)$$

where $\vec{s}_B$ and $\vec{r}_B$ is the Bloch vector of σ^*_B and ϱ, respectively. The real vector $\vec{g}_B$ is given by $(\varrho\circ g)_B =$

$\mathrm{Tr}_A(\varrho \circ g) = \frac{1}{2}\vec{g}_B \cdot \vec{\sigma}$ with $|i\rangle$ being eigenstates of σ^* ($\sigma^* = \sum_i \lambda_i |i\rangle\langle i|$), and the matrix g being

$$g_{ij} = \begin{cases} \frac{\lambda_i+\lambda_j}{2} \frac{\log\lambda_i - \log\lambda_j}{\lambda_i-\lambda_j} - 1 & \text{for } \lambda_i \neq \lambda_j \\ 0 & \text{for } \lambda_i = \lambda_j \end{cases} \qquad (6)$$

In this way, it can be seen that the local property of σ^* is strongly related to the extremal condition with respect to the local filtering.

The above discussion can be extended to any system in a very straightforward manner. For the multi-party system, the local filtering of the type $I\otimes \ldots \otimes e^{t\vec{n}\cdot\vec{\sigma}/2} \otimes \ldots \otimes I$ can be applied. For the party with d-dimension, the set of Pauli matrices is replaced with the set of d^2-1 Hermitian generators $\vec{J}$ of SU(d). Then we arrive at the main result of this paper:

Let ϱ be an entangled state in any multi-party system with any dimension. The reduction of the closest disentangled (and PPT) state σ^* with respect to the party X must satisfy $\vec{s}_X = \vec{r}_X + \vec{g}_X$, where $\vec{s}_X$ and $\vec{r}_X$ are the generalized Bloch vector of σ^*_X and ϱ_X, respectively, and $(\varrho \circ g)_X = \frac{1}{2}\vec{g}_X \cdot \vec{J}$.

It has been proved in Ref. [4], if $E_R(\varrho) = \max\{S(\varrho_A) - S(\varrho), S(\varrho_B) - S(\varrho)\}$, σ^* must have the same reduction as ϱ. According to the above, the necessary and sufficient condition for which the reductions are the same to each other is given by $\vec{g}_X = 0$. If σ^* commutes with ϱ, σ^* is diagonalized in the same basis as ϱ. In this case $\varrho \circ g = 0$, and hence $\vec{g}_X = 0$ for every party X. Therefore, $[\varrho, \sigma^*] = 0$ is a sufficient condition for which the reductions are the same to each other.

Now it is worth to check how the condition of $\vec{s}_X = \vec{r}_X + \vec{g}_X$ is satisfied in analytically solved examples of the relative entropy of entanglement. In all of the already solved examples, it can be seen that $\vec{g}_X = 0$ and the reductions are the same to each other. Does σ^* commute with ϱ in all examples? The answer is no. Instead, we found that all examples satisfy a condition, which is sufficient for $(\varrho \circ g)_A = (\varrho \circ g)_B = 0$ but weaker than $[\varrho, \sigma^*] = 0$, that is

$$(|j\rangle\langle j|[\varrho, \sigma^*]|i\rangle\langle i|)_A = (|j\rangle\langle j|[\varrho, \sigma^*]|i\rangle\langle i|)_B = 0 \qquad (7)$$

for any i and j. Here, $[A, B] \equiv AB - BA$, and $|i\rangle$'s are the eigenstates of σ^*. Depending on how to satisfy the condition, the examples are mainly classified in the following two categories:

(i) $[\varrho, \sigma^*] = 0$ and Eq. (7) is satisfied. The Bell diagonal states in two qubits [1], maximally entangled mixed states in two qubits [2,5], and isotropic state with any dimension [3] belong to this category.

(ii) In the support space of ϱ, $(|j\rangle\langle i|)_A = (|j\rangle\langle i|)_B = 0$ for all $i \neq j$, and Eq. (7) is satisfied. The maximally correlated states (including pure states) [3,6] and the state proposed in Ref. [7] belong to this category.

It is interesting to note that, if we wash out the classical correlations as well as the quantum correlations, the closest "uncorrelated" state is $\sigma_u = \varrho_A \otimes \varrho_B \otimes \varrho_C \cdots$ [1], where the reductions of σ_u are always the same as ϱ. In the case of the closest disentangled state, although there is no guarantee that the reductions are the same, $(\varrho \circ g)_X = 0$ is rather widely satisfied and reductions are the same in many cases as shown above.

Instead of the local filtering, we can consider the local unitary transformation as follows:

$$\sigma' = (I \otimes e^{it\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I \otimes e^{-it\vec{n}\cdot\vec{\sigma}/2}), \qquad (8)$$

which also ensures that σ' is disentangled (or PPT) for any t. The same discussion as the local filtering case gives

$$([\varrho, \log\sigma^*])_B = \frac{i}{2}\vec{h}_B \cdot \vec{\sigma} = 0, \qquad (9)$$

with $\vec{h}_B$ being a real vector. Therefore, the closest disentangled (and PPT) state must satisfy both Eq. (5) and Eq. (9) and Alice's counterparts. It is interesting to note that, even though $S(\varrho||\sigma^*) \neq S(\varrho||\sigma^*_{PPT})$ where σ^* and σ^*_{PPT} is the closest disentangled and PPT state of ϱ, respectively, both σ^* and σ^*_{PPT} satisfy the same equations of (5) and (9) (and Alice's counterparts). The total number of these equations in the $d \otimes d$ bipartite system is $4(d^2-1)$. Therefore, *in principal*, d^4-1 independent parameters in σ^* can be reduced to d^4-4d^2+3 by solving those equations. Although both Eq. (5) and Eq. (9) are not still tractable, the number of the remaining parameters is only three in the case of the simplest $2 \otimes 2$ systems.

To conclude, we study the extremal condition with respect to the local filtering and obtained the set of equations both the closest disentangled and PPT state must satisfy without explicitly using the condition that σ^* must be disentangled (or PPT). We showed that the local property of σ^* is strongly related to the extremal condition of the local filtering. Further, we obtained the sufficient condition for which σ^* has the same reduction as the given entangled state ϱ, and showed that the condition has been rather widely satisfied.

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[1] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[2] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[3] E. M. Rains, Phys. Rev. A **60**, 179 (1999).
[4] M. B. Plenio, S. Virmani, and P. Papadopoulos, J. Phys. A: Math. Gen. **33**, L193 (2000).
[5] F. Verstraete, K. Audenaert, and B. DeMoor, Phys. Rev. A **64**, 012316 (2001).
[6] S. Wu and Y. Zhang, quant-ph/0004018 .
[7] J. Eisert, T. Felbinger, P. Papadopoulos, M. B. Plenio, and M. Wilkens, Phys. Rev. Lett. **84**, 1611 (2000).