# The Maximum Expectation of Bell's Operators and Confirming Multipartite Entanglement

Koji Nagata, Masato Koashi, and Nobuyuki Imoto
*CREST Research Team for Interacting Carrier Electronics, School of Advanced Sciences, The Graduate University for Advanced Studies (SOKENDAI), Hayama, Kanagawa, 240-0193, Japan*

Since 1980's, it has been a problem how to confirm multipartite entanglement experimentally. Recently, we have been given precious experimental data by efforts of experimentalists [1,2]. Proper analysis of these experimental data then becomes necessary, and as a result of such analysis [3], the experimental data obtained by Pan *et al.* [2] confirm the existence of genuinely three-particle entanglement. However it was discussed that for other experimental data there is a loophole problem in insisting three-particle entanglement, and the loophole problem remains unresolved [4]. This means that there have been not enough discussions about what kind of data are at least needed for our purpose.

There have been many researches on the problem that provide inequalities for functions of experimental correlations [3–10]. Among them, much attention has been paid to the expectations of Bell-Mermin operators [11] for hybrid separable-inseparable states. The hybrid separable-inseparable states are depicted as follows. Consider a partition of $n$-particle system $\{1,2,\ldots,n\}$ into $k$ nonempty and disjoint subsets $\alpha_1,\ldots,\alpha_k$, where $\sum_{i=1}^{k}|\alpha_i|=n$, to which we refer as a $k$-partite split of the system [12]. Let us now consider the density operators $\rho$ on $\mathcal{H}=\bigotimes_{j=1}^{n}\mathcal{H}_j$, where $\mathcal{H}_j$ represents the Hilbert space with respect to particle $j$. Then all hybrid separable-inseparable states with respect to partition $\alpha_1,\ldots,\alpha_k$ can be written as

$$\rho=\sum_l p_l\left(\otimes_{i=1}^{k}\rho_l^{\alpha_i}\right),(p_l\geq 0,\sum_l p_l=1), \tag{1}$$

where $\rho_l^{\alpha_i},\forall l$ are the density operators on the partial Hilbert space $\bigotimes_{j\in\alpha_i}\mathcal{H}_j$. States (1) are called $k$-separable with respect to the partition $\alpha_1,\ldots,\alpha_k$.

Assuming $k$-partite split of the system without assuming a specific partition, Werner and Wolf derived an upper bound $2^{(n-k)/2}$ for expectation values of $n$-particle Bell-Mermin operators under the assumption that suitable partial transposes of the density operator are positive [10]. The inequality derived by Werner *et al.* is indeed useful because we can determine the minimum of $k$ such that the given state is not $k$-separable [13]. Recently, Uffink presented an upper bound of quadratic inequalities for partitions of the form $\{1\},\{2,3,\ldots,n\}$ as tests for $n$-qubit entanglement [8].

We shall determine the optimal upper bound of expectations of Bell-Mermin operators for any partition of the systems $\alpha_1,\ldots,\alpha_k$. It turns out that this maximum depends only on two parameters $k$ and $m$, and not on the detailed configuration of the partition, where $m$ is the number of particles which are not entangled with any other particles. The maximum is $2^{(n+m-2k+1)/2}$, except for the case that the system is fully separable. We also determine the optimal upper bound of the quadratic inequality proposed by Uffink for any partition of the systems.

If we impose a condition on the observables, the optimal upper bound becomes as small as $2^{(n-2k+1)/2}$. It is then shown that we can use a quadratic inequality stronger than Uffink's quadratic inequalities presented in Ref. [8] as tests for multipartite entanglement in correlation experiments.

In what follows, we determine the optimal upper bounds of the expectations of Bell-Mermin operators for hybrid separable-inseparable states with respect to partition $\alpha_1,\ldots,\alpha_k$. It is assumed that a measurement with two outcomes, $\pm 1$, is performed on each particle. Such a measurement is generally described by a positive-operator-valued measure (POVM), $\{F_+,F_-\},F_++F_-=\mathbf{1},F_+,F_-\geq 0$, and the corresponding observable is given by a Hermitian operator $A=F_+-F_-$, which has a spectrum in $[-1,1]$. We assume that for each particle $j$, either of two such observables $A_j$ or $A'_j$ is chosen, where $-\mathbf{1}\leq A_j,A'_j\leq\mathbf{1},\forall j$.

Let $f(x,y)$ denote a function $\frac{1}{\sqrt{2}}e^{-i\pi/4}(x+iy),x,y\in\mathbf{R}$. Note that this function is invertible, as $x=\Re f-\Im f,y=\Re f+\Im f$. The Bell-Mermin operators $\mathcal{B}_{\mathbf{N}_n}$ and $\mathcal{B}'_{\mathbf{N}_n}$ are defined by [10,11]

$$f(\mathcal{B}_{\mathbf{N}_n},\mathcal{B}'_{\mathbf{N}_n})=\otimes_{j=1}^{n}f(A_j,A'_j), \tag{2}$$

where $\mathbf{N}_n=\{1,2,\ldots,n\}$. We also define $\mathcal{B}_\alpha$ for any subset $\alpha\subset\mathbf{N}_n$ by

$$f(\mathcal{B}_\alpha,\mathcal{B}'_\alpha)=\otimes_{j\in\alpha}f(A_j,A'_j). \tag{3}$$

Without loss of generality, we assume that $|\alpha_i|=1$ for $i\in\mathbf{N}_m$ and $|\alpha_i|\geq 2$ for $i\in\mathbf{N}_k\backslash\mathbf{N}_m$.

With the above notations, we can derive, after some theoretical calculations, the result that, for any state $\rho$ which is $k$-separable with respect to $\alpha_1,\ldots,\alpha_k$,

$$(\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}])^2+(\mathrm{tr}[\rho\mathcal{B}'_{\mathbf{N}_n}])^2\leq 2^{n+m-2k+1}. \tag{4}$$

This inequality also implies

$$|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\leq 2^{(n+m-2k+1)/2}. \tag{5}$$

It is known that $|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\leq 1$ when the system is fully separable [10]. Hence we obtain an upper bound

$$|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\leq\begin{cases}2^{(n+m-2k+1)/2} & k<n\\ 1 & k=n.\end{cases} \tag{6}$$

The equality of the relation (6) holds when $\langle \mathcal{B}_{\alpha_i} \rangle = \langle \mathcal{B}'_{\alpha_i} \rangle = 1$ for $i \in \mathbf{N}_m$, $\langle \mathcal{B}_{\alpha_i} \rangle = \langle \mathcal{B}'_{\alpha_i} \rangle = 2^{(|\alpha_i|-2)/2}$ for $i \in \mathbf{N}_{k-1} \backslash \mathbf{N}_m$, and $\langle \mathcal{B}_{\alpha_k} \rangle = 2^{(|\alpha_k|-1)/2}$. We can find a state and Hermitian operators $-\mathbf{1} \le A_j, A'_j \le \mathbf{1}$ that satisfy the above relations [14]. Hence the bound (6) is optimal. The maximum depends only on two parameters $k$ and $m$ but not on the detailed configuration of the partition.

For partitions of the form $\{1\}, \{2\}, \ldots, \{m\}, \{m+1, \ldots, n\}$, the relation (6) leads to the result of Gisin and Bechmann-Pasquinucci [6], i.e., the bound $|\langle \mathcal{B}_{\mathbf{N}_n} \rangle| \le 2^{(n-m-1)/2} (m \le n-1)$. Noting that $m \le k-1$ when $k < n$, the relation (6) leads to the result of Werner and Wolf [10], i.e., $|\langle \mathcal{B}_{\mathbf{N}_n} \rangle| \le 2^{(n-k)/2}$ by taking the maximum over $m$ with fixed $k$. Collins *et al.* considered the cases for partitions of the form $\{1\}, \{2\}, \{3, 4\}$ or $\{1, 2\}, \{3, 4\}$ or $\{1\}, \{2, 3, 4\}$ and presented the bounds as $\sqrt{2}, \sqrt{2}, 2$, respectively [7]. These bounds are also derived from the relation (6). In Ref. [8], Uffink considered the case for partitions of the form $\{1\}, \{2, 3, \ldots, n\}$, and has presented a following quadratic inequality for testing whether $n$-qubit states are fully entangled as

$$\langle \mathcal{B}_{\mathbf{N}_n} \rangle^2 + \langle \mathcal{B}'_{\mathbf{N}_n} \rangle^2 \le 2^{n-2}. \quad (7)$$

Actually, we can see that the relation (7) can be derived not only for multiqubit systems but also for any multipartite system. In what we should pay attention to, we have to check that for all partitions of the form $\{1, 2, \ldots, r\}, \{r+1, \ldots, n\} (1 \le r \le n)$, the optimal upper bounds are smaller than or equal to $2^{n-2}$, in order to see that the relation (7) can be used as tests for $n$-particle entanglement. By means of the relation (4), we can prove that the violations of the relation (7) are indeed a sufficient for confirming $n$-particle entangled states.

So far, we have shown the maximal values of correlation functions over all observables satisfying $-\mathbf{1} \le A_j, A'_j \le \mathbf{1}$. On the other hand, there may be the situations where we want to test the multiparticle entanglement on the assumption that we are sure about what observables are measured in the experiment. In what follows, we show that quite strong inequalities are obtained if we impose the condition $\{A_j, A'_j\} = \mathbf{0}$, which is, for example, satisfied by a familiar choice $A_j = \sigma_x^j$ and $A_j = \sigma_y^j$. We can obtain the following proposition:

*Proposition.*— Under the conditions $-\mathbf{1} \le A_j, A'_j \le \mathbf{1}$, $\{A_j, A'_j\} = \mathbf{0}, \forall j$, and that $\rho$ is $k$-separable for partition $\alpha_1, \alpha_2, \ldots, \alpha_k$, the maximum value of $|\mathrm{tr}[\rho \mathcal{B}_{\mathbf{N}_n}]|$ over $\rho$, $A_j$ and $A'_j$ is $2^{(n-2k+1)/2}$.

We also get the following:

$$(\mathrm{tr}[\rho \mathcal{B}_{\mathbf{N}_n}])^2 + (\mathrm{tr}[\rho \mathcal{B}_{\mathbf{N}_n}])^2 \le 2^{n-2k+1}. \quad (8)$$

From the relation (8), for $k = 2$ we get an inequality as tests for $n$-particle entanglement by

$$\langle \mathcal{B}_{\mathbf{N}_n} \rangle^2 + \langle \mathcal{B}'_{\mathbf{N}_n} \rangle^2 \le 2^{n-3}. \quad (9)$$

Hence if the measurement setups are chosen as $\{A_j, A'_j\} = \mathbf{0}, \forall j$, then we can use the stronger inequality as tests for multipartite entanglement in comparison with the relation (7).

In summary, we have derived the optimal upper bound of the expectations of $n$-particle Bell-Mermin operators assuming that $n$-particle density operators are $k$-separable, $\alpha_1, \alpha_2, \ldots, \alpha_k$, by $2^{(n+m-2k+1)/2}$, where $m$ is the number of particles which are not entangled with any other particles, except for the case that the system is fully separable. If we impose a condition on the observables, then the optimal upper bound becomes $2^{(n-2k+1)/2}$ and then, we have derived inequalities stronger than those obtained by Uffink as tests for multipartite entanglement in correlation experiments.

---


[1] D. Bouwmeester *et al.*, Phys. Rev. Lett. **82**, 1345 (1999); C. A. Sackett *et al.*, Nature (London) **404**, 256 (2000); A. Rauschenbeutel *et al.*, Science **288**, 2024 (2000); J. -W. Pan *et al.*, Phys. Rev. Lett. **86**, 4435 (2001).

[2] J. -W. Pan *et al.*, Nature (London) **403**, 515 (2000).

[3] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. A **65**, 042314 (2002).

[4] M. Seevinck and J. Uffink, Phys. Rev. A **65**, 012107 (2002).

[5] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).

[6] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).

[7] D. Collins *et al.*, Phys. Rev. Lett. **88**, 170405 (2002).

[8] J. Uffink, Phys. Rev. Lett. **88**, 230406 (2002).

[9] M. Seevinck and G. Svetlichny, Phys. Rev. Lett. **89**, 060401 (2002).

[10] R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000).

[11] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).

[12] W. Dür and J. I. Cirac, Phys. Rev. A **61**, 042314 (2000).

[13] We can estimate the minimum number of entangled particles as $n/k, (n/k \in \mathbf{N})$ or $[n/k] + 1, (n/k \notin \mathbf{N})$, when we are given the maximum positive integer $k$ which is allowed by experimental data.

[14] First consider qubit systems and find an example of a state $\rho$ and observables $A, A'$. Next consider a state $\rho \oplus \mathbf{0}$ and observables $A \oplus \mathbf{0}, A' \oplus \mathbf{0}$.