

High Precision measurement of weak forces using Schrödinger Cat resources

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Non classical states of light have received considerable attention in the field of quantum and atom optics. Many non-classical states of light have been experimentally produced and characterised. There are a number of suggested, and actual, applications of these states in quantum information processing including; quantum computation, communication and cryptography to name but a few. They have also been proposed for high precision measurements which include improving the sensitivity of Ramsey fringe interferometry[1] and the detection of weak tidal forces due to gravitational radiation. In this article we consider how non classical states of simple harmonic oscillators may be used to improve the detection sensitivity of weak classical forces.

Let us begin by establishing the classical limit for force detection. When a classical force, $F(t)$, acts for a fixed time on a simple harmonic oscillator, it displaces the complex amplitude of the oscillator in phase space with the amplitude and phase of the displacement determined by the time dependence of the force. The action of the force in the interaction picture is simply represented by the unitary displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (1)$$

where a, a^\dagger are the annihilation and creation operators for the single mode of the electromagnetic field satisfying $[a, a^\dagger] = 1$, and α is a complex amplitude which determines the average field amplitude, $\langle a \rangle = \alpha$. For simplicity we will assume that the force displaces the oscillator in a phase space direction that is orthogonal to the coherent amplitude of the initial state. To detect the force we would need to measure the signal of the quadrature $\hat{Y} = -i(a - a^\dagger)$. If the oscillator begins in a coherent state $|\alpha_0\rangle$, (α_0 is real) the displacement $D(i\epsilon)$ causes the coherent state to evolve to $e^{i\epsilon\alpha_0}|\alpha_0 + i\epsilon\rangle$. The signal to noise ratio is then $SNR = S/\sqrt{V} = 2\epsilon$ which must be greater than unity to be resolved. This establishes the standard quantum limit (SQL) of $\epsilon_{SQL} \geq 1/2$.

It is well known[2] that this limit may be overcome if the oscillator is first prepared in a squeezed state (a uniquely quantum mechanical state) for which the uncertainty in the momentum quadrature is reduced below the coherent state level. For a momentum squeezed state (with mean photon number n_{tot}) it is straightforward to show the minimum detectable displacement is $\epsilon_{min} \geq 1/\sqrt{4n_{tot}}$. This scales as the inverse square root of the mean photon number and means the only way to improve the sensitivity is to increase n_{tot} . Can we do better using different resources.

We now consider another class of non classical states, based on a coherent superposition of coherent states, which can be entangled over N modes. The N mode entangled cat state has the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha, \alpha, \dots, \alpha\rangle + |-\alpha, -\alpha, \dots, -\alpha\rangle) \quad (2)$$

where n_{tot} is the total photon number of the entire state. Parkins and Larsabal[?] recently suggested how this highly entangled state might be formed in the context of cavity QED and quantised motion of a trapped atom or ion. The weak force $D(i\epsilon)$ (for $\epsilon \ll 1$) acts simultaneously on all modes of this N party entangled state. The resulting state is

$$|\psi(\theta)\rangle = \frac{e^{iN\theta}}{\sqrt{2}}|\alpha + i\epsilon, \dots, \alpha + i\epsilon\rangle + \frac{e^{-iN\theta}}{\sqrt{2}}|-\alpha + i\epsilon, \dots, -\alpha + i\epsilon\rangle$$

where $\theta = \epsilon\alpha$. The theory of optimal parameter estimation indicates that the limit on the precision with which the displacement parameter be estimated with $n_{tot} \gg 1$ is bounded by

$$\epsilon_{min} = \frac{1}{\sqrt{N[1+4n_{tot}]}} \sim \frac{1}{\sqrt{4Nn_{tot}}}$$

and realises the Heisenberg Limit. Let us now compare this to the case where one has N copies of the single mode cat state $|\alpha\rangle + |-\alpha\rangle$. In this case (for $\alpha^2 \gg 1$) the minimum detectable displacement is $\epsilon_{min} = 1/\sqrt{[N+4n_{tot}]}$ while a single mode cat state with mean photon number n_{tot} achieves $\epsilon_{min} = 1/\sqrt{[1+4n_{tot}]}$. Now how do we interpret these results. For large n_{tot} (a regime we need to be in to achieve a good sensitivity) we find the N mode entangled cat state has an extra critically important \sqrt{N} improvement over N individual copies of a single mode cat state (both states have the same mean photon number). This is due to the entangled cat state and the collective (Bell like) measurement. These results indicate that it would seem to be most efficient for a large fixed n_{tot} to have as many modes in the entangled state as possible.

It is enlightening to compare our results to the study of Ramsey fringe interferometry introduced by Bollinger et al. [3] and discussed by Huelga et al.[1]. In Ramsey fringe interferometry the objective is to detect the relative phase difference between two superposed states, $\{|0\rangle, |1\rangle\}$. The theory of quantum parameter estimation[4] indicates in this case that we should choose the input state as $|\psi\rangle_i = (|0\rangle + |1\rangle)/\sqrt{2}$ and the optimal measurement is a projective measurement in the basis $|\pm\rangle = |0\rangle \pm |1\rangle$. The probability to obtain the result $+$ is $P(+|\theta) = \cos^2\theta$. In N repetitions of the measurement the uncertainty in the inferred parameter is $\delta\theta = 1/\sqrt{N}$ which achieves the lower classical bound for quantum phase parameter estimation. However it was first noted by Bollinger et al.[3] that a more effective way to use the N level systems is to first prepare them in the maximally entangled state.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2 \dots |0\rangle_N + |1\rangle_1|1\rangle_2 \dots |1\rangle_N) \quad (3)$$

In this case parameter estimation gives the Heisenberg lower bound of $\delta\theta = 1/N$. The Hilbert space of N two level systems is the tensor product space of dimension 2^N . The entangled state in Eq.(3) however resides in a lower dimensional subspace of permutation symmetric states[5] which can be written in the form $|\psi\rangle = \frac{1}{\sqrt{2}}(|-N/2\rangle_{N/2} + |N/2\rangle_{N/2})$ and so we can regard the state as an SU(2) ‘cat state’ for N two-level atoms. Hence it is straightforward to see that a single N level atom can achieve the same frequency sensitivity. Their equivalence can be also be understood by noting that the sensitivity of such frequency measurement is proportional to the energy difference of the states involved. What entanglement allows is for one to create an effective state without the need of resorting to create a superposition between certain ground state and a highly excited one.

To conclude, we have in this article shown how superpositions of coherent states can be used to achieve extremely sensitive force detection.

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