QUANTUM RANDOM WALKS ON $\mathbf{Z}^d$

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We define and consider quantum random walks in higher dimensions as in the case of one dimension studied by Konno, Namiki and Soshi (quant-ph/0205065 (2002)).

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The classical random walk (CRW) on the line is a well-studied process. It plays an essential role in various fields of solid-state physics, polymer chemistry, biology, astronomy, mathematics and computer science.

The time evolution of the CRW on $\mathbf{Z}$ is given by the following way. At each step particle moves one position left or right with probability p or q respectively, where $\mathbf{Z}$ is the collection of integers. After n step, the probability $P_k(n)$ that the particle at position k at time n satisfies

$$P_k(n+1) = pP_{k+1}(n) + qP_{k-1}(n)$$

In particular, if $p = q = 1/2$, then the CRW is called symmetric.

In this paper, we consider a quantum variation of the CRW called quantum random wallk (QRW) on $\mathbf{Z}^d$. Recent years, QRWs have been investigated by many reseachers, for examples, Aharonov *et al.*[1], Ambainis *et al.*[2], Ambainis *et al.*[3], Bach *et al.*[4], Dür *et al.*[5], Kempe [6], Konno [7,8], Konno, Namiki and Soshi [9,17], Konno *et al.*[16], Mackay *et al.*[10], Moore and Russell [11], Nayak and Vishwanath [12], Travaglione and Milburn [14], Yamasaki, Kobayashi and Imai [15].

First, we consider a one-dimensional QRW (the Hadamard walk) whose time evolution is given by the following Hadamard transformation (see Nielsen and Chuang [13]):

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard matrix H is unitary. The Hadamard walk is a quantum generalization of a symmetric CRW in one dimension with an additional degree of freedom called the chirality. The chirality takes values left and right, and means the direction of the motion of the particle. The evolution of the Hadamard walk is given by the following rules. At each time step, if the particle has the left chirality, it moves one step to the left, and if it has the right chirality, it moves one step to the right.

More precisely, the Hadamard matrix H acts on two chirality states $|L\rangle$ and $|R\rangle$:

$$|L\rangle \;\to\; \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \qquad |R\rangle \;\to\; \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

where L and R refer to the right and left chirality state respectively. In fact, define

$$|L\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |R\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so we have

$$H|L\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \qquad H|R\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$$

We introduce P and Q matrices as follows:

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

with $H = P + Q$. Remark that P (resp. Q) represents the particle moves to the left (resp. right) with equal probability. Let $\Psi(k,n) = {}^t\,[\Psi_L(k,n), \Psi_R(k,n)]$ (t stands for the transposed operator) be the two component of probability amplitudes of the particle being at position k at time n, with the chirality left (upper component) or right (lower component). Then, the dynamics for Ψ is given by

$$\Psi(k, n+1) = P\Psi(k+1, n) + Q\Psi(k-1, n)$$

It should be noted that P and Q are useful tools in the study of iterates of H.

The set of initial qubit states is defined by

$$\Phi = \left\{ \varphi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbf{C}^2 : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

The symmetry of the probability distribution for the one-dimensional Hadamard walk depends heavily on the initial qubit state φ (see Konno *et al.* [8]).

An analysis of the one-dimensinal QRW can be extended to higher dimensions. We define a generalization of the Hadamard transformation as

$$\mathbf{H}_d = \overbrace{H \otimes H \otimes \cdots \otimes H}^{d}$$

where $\otimes$ stands for the tensor product. We consider the set of initial qubit states is given by

$$\Phi^{(d)} = \{ \varphi_1 \otimes \varphi_2 \otimes \cdots\cdots \otimes \varphi_d : \varphi_i \in \Phi \ (i = 1, 2, \cdots, d) \},$$

that is, $\Phi^{(d)}$ is d-fold product space of Φ.

From now on, we discuss the two-dimensional Hadamard walk. The definition of $\mathbf{H}_d$ implies

$$\mathbf{H}_2 = H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

As in the one-dimentional case, we introduce P_1, P_2, P_3 and P_4 matrices:

$$P_1 \equiv \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = P \otimes P,$$

$$P_2 \equiv \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = P \otimes Q,$$

$$P_3 \equiv \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Q \otimes P,$$

$$P_4 \equiv \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} = Q \otimes Q.$$

We assume that P_1 represents the particle moves to the up. Similarly, P_2, P_3 and P_4 represent the particle moves to the right, left and down, respectively. Each P_i $(i = 1, 2, 3, 4)$ expresses the particle moves to one of four directions with equal probability. Then, we can calculate probability amplitude $\Psi_n^{\varphi}(k, l)$ at position $(k, l) \in \mathbf{Z}^2$ of the Hadamard walk X_n^{φ} at time n starting from initial qubit state $\varphi \in \Phi^{(2)}$ at position $(0, 0)$.

For instance, we consider $\Psi_2^{\varphi}(i, j)$ and $P(X_2^{\varphi} = (i, j))$ $(i, j = 0, 1, 2)$. Then we get the probability amplitude $\Psi_2^{\varphi}(i, j)$ and the probability $P(X_2^{\varphi} = (i, j))$ $(i, j = 0, 1, 2)$ as follows:

$$\begin{aligned} \Psi_2^{\varphi}(0, 2) &= (P_1 P_1)\varphi, \quad \Psi_2^{\varphi}(2, 0) = (P_2 P_2)\varphi, \\ \Psi_2^{\varphi}(-2, 0) &= (P_3 P_3)\varphi, \quad \Psi_2^{\varphi}(0, -2) = (P_4 P_4)\varphi, \\ \Psi_2^{\varphi}(1, 1) &= (P_1 P_2 + P_2 P_1)\varphi, \\ \Psi_2^{\varphi}(-1, 1) &= (P_1 P_3 + P_3 P_1)\varphi, \\ \Psi_2^{\varphi}(1, -1) &= (P_2 P_4 + P_4 P_2)\varphi, \\ \Psi_2^{\varphi}(-1, -1) &= (P_3 P_4 + P_4 P_3)\varphi, \\ \Psi_2^{\varphi}(0, 0) &= (P_1 P_4 + P_4 P_1 + P_2 P_3 + P_3 P_2)\varphi \end{aligned}$$

Using $\Psi_2^{\varphi}(i, j)$, we obtain the probability distribution at time step 2:

$$P(X_2^{\varphi} = (i, j)) = |\Psi_2^{\varphi}(i, j)|^2 = \Psi_2^{\varphi}(i, j)^* \Psi_2^{\varphi}(i, j)$$

where * means the adjoint operator.

At poster session, I discuss necessary and sufficient conditions of symmetry of probability distribution for the d-dimensional QRW as in the one-dimensional case studied by Konno, Namiki and Soshi [9].

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