Some remarks on generating functions for absorbing probability of quantum random walks

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Abstract

The hitting probability problem of quantum random walks with absorbing boundaries is discussed and with the path method the explicit formulae of the generating functions for the amplitudes of them are shown.

Keywords: *Quantum random walks, Hadamard walk, first hitting probability.*

1 Introduction

Recently the study of quantum random walks is widely developed ([1, 2, 3, 4, 5, 6, 7]). Though differences between classical random walks and quantum walks exist, the notion of "path" is still valid to analyze certain features of quantum walks. At first the definition of path is introduced briefly. Secondly we show the simple method using path to obtain the generating functions for absorbing boundaries problem.

As shown in [4, 5, 6] each path of random walks corresponds directly to that of quantum walks. Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a unitary 2×2 matrix and set $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$. Since these matrices represent the action of quantum walks, the classical path of random walks corresponds to the product of P and Q.

To study the combinations of P and Q we need $R = PQ$ and $S = QP$. We obtain Table 1 of computations P, Q, R and S in the case of Hadamard matrix.

	P	Q	R	S
P	P	R	R	P
Q	S	$-Q$	Q	$-S$
R	P	$-R$	R	$-P$
S	S	Q	Q	S

Table 1: P, Q, R and S generate the group for Hadamard walk. The factor $1/\sqrt{2}$ is neglected.

By such table, we can write down the certain probability amplitude with the linear combinations of P, Q, R, S, and from the fact that they are linearly independent all we have to do is to determine the coefficients of P, Q, R, S for certain set of paths.

2 Generating functions for absorbing problems

In this section we study the generating functions for absorbing problems. Though the results are partially included in [2], the method we use here makes simple and straightforward interpretation from classical random walks to quantum walks.

Let $x > 0$ and think about Hadamard walk (for simplify) starting at x. We consider the first hitting amplitude and probability at 0. Set p_n, q_n, r_n and s_n be the coefficients of the first hitting paths starting at x by n-step.

It is clear that such paths have the form $P \ldots P$ or $P \ldots Q$, i.e. P must appear at the left. Therefore the problem becomes simple because $q_n = s_n = 0$ and all we have to consider is only the value of p_n and r_n. Observe the relation

$$\begin{pmatrix} p_n^{(x)} & r_n^{(x)} \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} p_{n-1}^{(x-1)} & r_{n-1}^{(x-1)} \\ 0 & 0 \end{pmatrix} UP + \begin{pmatrix} p_{n-1}^{(x+1)} & r_{n-1}^{(x+1)} \\ 0 & 0 \end{pmatrix} UQ$$

and $UP = P^*U, UQ = Q^*U$, we have

$$p_n^{(x)} = (p_{n-1}^{(x-1)} + r_{n-1}^{(x-1)})/\sqrt{2} \text{ and } r_n^{(x)} = (p_{n-1}^{(x+1)} - r_{n-1}^{(x+1)})/\sqrt{2}$$

for Hadamard walk. The generating functions $p_x(z) = \sum_{n=1}^{\infty} p_n^{(x)} z^n$ and $r_x(z) = \sum_{n=1}^{\infty} r_n^{(x)} z^n$ ($|z| \le 1$) satisfy the following form:

$$p_x(z) = \frac{z}{\sqrt{2}}(p_{x-1}(z) + r_{x-1}(z)) \ , \ \ r_x(z) = \frac{z}{\sqrt{2}}(p_{x+1}(z) - r_{x+1}(z)).$$

From this we have $p_x(z) - \sqrt{2}(z - 1/z)p_{x-1}(z) - p_{x-2}(z) = 0$ and the two solutions are $\lambda_{\pm} = \frac{z^2-1\pm\sqrt{z^4+1}}{\sqrt{2}z}$. Because the boundary conditions for $p_x(z)$ are $p_1(z) = z$ and $\lim_{x\to\infty} |p_x(z)| < \infty$, the explicit formula is the following:

$$p_x(z) = z\lambda_+^{x-1} \text{ and } r_x(z) = \frac{-1+\sqrt{z^4+1}}{z}\lambda_+^{x-1}.$$

Next we consider the two absorbing boundaries at 0 and N case. Suppose $p_x^{(N)}(z)$ and $r_x^{(N)}(z)$ satisfy

$$p_x^{(N)}(z) = A_z\lambda_+^{x-1} + B_z\lambda_-^{x-1} \ \ , \ \ r_x^{(N)}(z) = C_z\lambda_+^{x-N+1} + D_z\lambda_-^{x-N+1},$$

all we have to do is to determine the coefficients A_z, B_z, C_z, D_z by using boundary conditions: $p_1^{(N)}(z) = z$ and $r_{N-1}^{(N)}(z) = 0$. From the boundary conditions, we have $C_z + D_z = 0$ and $A_z + B_z = z$, so

$$p_x^{(N)}(z) = \left(\frac{z}{2} + E_z\right)\lambda_+^{x-1} + \left(\frac{z}{2} - E_z\right)\lambda_-^{x-1} \ \ , \ \ r_x^{(N)}(z) = C_z(\lambda_+^{x-N+1} - \lambda_-^{x-N+1})$$

where $E_z = A_z - z/2 = z/2 - B_z$. To obtain E_z and C_z, we use $r_1(z) = (p_2(z) - r_2(z))z/\sqrt{2}$ and $r_{N-2}(z) = (p_{N-1}(z) - r_{N-1}(z))z/\sqrt{2} = p_{N-1}(z)z/\sqrt{2}$. Therefore

$$C_z(\lambda_+ - \lambda_-) = \frac{z}{\sqrt{2}}\left\{\left(\frac{z}{2} + E_z\right)\lambda_+^{N-2} + \left(\frac{z}{2} - E_z\right)\lambda_-^{N-2}\right\}$$
$$C_z(\lambda_+^{N-2} - \lambda_-^{N-2}) = \frac{z}{\sqrt{2}}\left\{\left(\frac{z}{2} + E_z\right)(-1)^{N-1}\lambda_+ + \left(\frac{z}{2} - E_z\right)(-1)^{N-1}\lambda_- + C_z(\lambda_+^{N-3} - \lambda_-^{N-3})\right\}$$

Solving the above equations gives

$$\begin{aligned} C_z &= \frac{z^2}{\sqrt{2}}(-1)^{N-2}(\lambda_+^{N-3} - \lambda_-^{N-3}) \\ &\times \left\{(\lambda_+^{N-2} - \lambda_-^{N-2})^2 - \frac{z}{\sqrt{2}}(\lambda_+^{N-2} - \lambda_-^{N-2})(\lambda_+^{N-3} - \lambda_-^{N-3}) - (-1)^{N-3}(\lambda_+ - \lambda_-)^2\right\}^{-1} \\ E_z &= \frac{z}{2}\left[(-1)^{N-1}(\lambda_+^2 - \lambda_-^2) + (\lambda_+^{N-2} + \lambda_-^{N-2})(\frac{z}{\sqrt{2}}(\lambda_+^{N-3} - \lambda_-^{N-3}) - (\lambda_+^{N-2} - \lambda_-^{N-2}))\right] \\ &\times \left\{(\lambda_+^{N-2} - \lambda_-^{N-2})^2 - \frac{z}{\sqrt{2}}(\lambda_+^{N-2} - \lambda_-^{N-2})(\lambda_+^{N-3} - \lambda_-^{N-3}) - (-1)^{N-3}(\lambda_+ - \lambda_-)^2\right\}^{-1}. \end{aligned}$$

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