The Chapman-Robbins Type Lower Bound for the Quantum Estimation

Yoshiyuki Tsuda and Keiji Matsumoto

ERATO QCI, JST, 5-28-3-302, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan.

1 Introduction

The quantum estimation is a problem to estimate the physical state of a quantum system, whose framework is made by Helstrom [3]. In this problem, the true value of the parameter of a prepared physical system is estimated. A goodness of an estimator is defined and we can evaluate the goodness of estimators. For example, an estimator might be better if the mean is closer to the true value of the parameter, and the variance is smaller. If the mean coincides with the true value of the parameter, then the estimator is said to be unbiased, and this unbiasedness is one of the criteria for the goodness of estimators. Among unbiased estimators, an estimator whose variance is the smallest might be the best estimator. From this criterion, the lower bound of the variance of unbiased estimators is discussed. Helstrom [3] gave the Cramér-Rao Type Lower Bound of the variance of unbiased estimators based on a generalization of the logarithmic derivatives of the classical probability distributions to the quantum density operators. This argument is developed, for example, Yuen and Lax [6], Holevo [5], Hirota [4] and many other authors. Their arguments assume that the density operator has a kind of logarithmic derivatives by the parameter, in this sense, their model is smooth or regular. However, there are some physical models in which the model is not smooth or regular. For example, the concurrence of two quantum bits are not regular since it has a singular point at which the value of concurrence can not be differentiated. In such a case, we have to generalize the arguments of Cramér-Rao type bound to the non-regular case. In the classical estimation problem, Hammersley [2] and Chapman and Robbins [1] proposed a lower bound for the non-regular case. Their arguments use the difference in stead for the derivative, and hence they do not assume the smoothness or regularity of the model. Moreover, if the model is smooth, their bound is equal to the Cramér-Rao type bound.

In this paper, we will give a Hammersley-Chapman-Robbins lower bound for the non-regular quantum estimation problem.

2 Problem

Let Θ be a subset of the p-dimensional Euclidean space $\boldsymbol{R}^p$. Let $g : \boldsymbol{R}^p \to \boldsymbol{R}^q$ be a map. Let $\mathcal{H}$ be a d-dimensional Hilbert space describing a quantum system. Let $\mathcal{L}$ be the set of linear operators on $\mathcal{H}$, and let $\mathcal{L}_H$ be the set of Hermitian operators in $\mathcal{L}$. A Hermitian operator $A \in \mathcal{L}_H$ describes an observable of the quantum system. Let $\mathcal{S}$ be the set of density operators in $\mathcal{L}_H$, that is, the trace $\mathrm{Tr}_{\mathcal{H}}\tau$ on $\mathcal{H}$ of any $\tau \in \mathcal{S}$ is one. The density operator $\tau \in \mathcal{S}$ denotes the physical state of the quantum system. A map $\rho : \Theta \to \mathcal{S}$ is given, and let ρ_θ denotes $\rho(\theta)$ for $\theta \in \Theta$. Suppose that, for $\theta_0 \in \Theta$, the physical state ρ_{θ_0} is prepared and we know that the state is one of elements of the set $\{\rho_\theta\}$ $(\theta \in \Theta)$, but we do not know the true value θ_0 of the parameter $\theta \in \Theta$. Our problem is to estimate $g(\theta)$ by observing the system. We are allowed to use any observable that is described by a Hermitian operator in $\mathcal{L}_H$. For $X \in \mathcal{L}_H$, let $\sigma(X)$ denotes the set of eigenvalues of X. Let $g(\Theta)$ be the set $\{g(\theta) \mid \theta \in \Theta\}$ and let $h = (h_1, ..., h_q)^\dagger$ be a map from $\sigma(X)$ to $g(\Theta)$. Define $\hat{g} = (\hat{g}_1, ..., \hat{g}_q)^\dagger$ by $h(X)$ where, for a normal operator Y with the spectral decomposition $Y = \sum_j \lambda_j \Pi_j$ and for any map f of $\{\lambda_j\}$, $f(Y)$ means $\sum_j f(\lambda_j)\Pi_j$. We call $\hat{g}$ an estimator of $g(\theta)$. Define a mean vector $\mu = (\mu_1, ..., \mu_q)^\dagger$ of $\hat{g}$ by

$$\mu_j = \mathrm{Tr}_{\mathcal{H}}(\rho_\theta \hat{g}_j)$$

and define a covariance matrix $V = (V_{j,k})_{j,k}$ of $\hat{g}$ by

$$V_{j,k} = \mathrm{Tr}_{\mathcal{H}}(\rho_\theta(\hat{g}_j - \mu_j)(\hat{g}_k - \mu_k))$$

If $\mu = g(\theta)$, then $\hat{g}$ is said to be unbiased at θ. If $\mu = g(\theta)$ for any $\theta \in \Theta$, then $\hat{g}$ is said to be uniformly unbiased, or simply, unbiased.

Our problem is to give a lower bound of the covariance matrix of an unbiased estimator.

If the model is regular, that is, ρ_θ and $g(\theta)$ are continuously differentiable by θ, then the Cramér-Rao type bound based on the symmetric logarithmic derivative (SLD) and the right logarithmic derivative (RLD) gives the lower bound (see, for example, [3, 5]). However, if the model is non-regular, that is, ρ_θ or $g(\theta)$ is not continuously differentiable by θ, then the arguments of logarithmic derivative may not be applied. In such a case, the ϵ-difference

$$\Delta_{x,\epsilon} f(x) \ = \ \frac{f(x+\epsilon) - f(x)}{\parallel \epsilon \parallel}$$

of a function f at x is useful instead of the derivative of f at x.

In the classical estimation problem, a lower bound of unbiased estimators for the non-regular model is given by Hammersley [2] and Chapman and Robbins [1]. In this paper, we generalize the Hammersley-Chapman-Robbins inequality to the quantum estimation problem.

3 Hammersley-Chapman-Robbins type bound

Let G be a $r \times q$ matrix whose (j,k)-component $G_{j,k}$ is defined by

$$\Delta_{\theta,\epsilon_j} g_k(\theta),$$

where ϵ_j are p-dimensional real vectors. Based on these ϵ_j, define an operator L_j by

$$\Delta_{\theta,\epsilon_j} \rho_\theta \ = \ \frac{\rho_\theta L_j + L_j^\dagger \rho_\theta}{2}.$$

Let J be a matrix whose (j,k)-component $J_{j,k}$ is defined by

$$J_{j,k} \ = \ \mathrm{Re}(\mathrm{Tr}_{\mathcal{H}}(\rho_\theta L_j L_k^\dagger)).$$

Then, we have the quantum version of the Hammersley-Chapman-Robbins inequality as follows.

Theorem *If an estimator $\hat{g}$ of $g(\theta)$ is unbiased, then the covariance matrix V is bounded by*

$$V \ \geq \ G^\dagger J G. \tag{1}$$

The inequality (1) holds for arbitrarily chosen ϵ_j. It implies that the supremum of $G^\dagger J G$ with respect to the choice of ϵ_j gives the most strict lower bound of this form.

References

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